Signals and Systems-23EC02

UNIT-II

Fourier Series & Fourier Transform

B.Tech., III-Sem., ECE

1. Fourier Series Representation of Continuous Time Periodic Signals
2. Dirichlet's Conditions
3. Trigonometric Fourier Series
4. Exponential Fourier Series
5. Relation between Trigonometric and Exponential Fourier Series
6. Properties of Fourier Series
7. Complex Fourier Spectrum
8. Deriving Fourier Transform from Fourier Series
9. Fourier Transform of Arbitrary Signals
10. Fourier Transform of Standard Signals
11. Fourier Transform of Periodic Signals
12. Introduction to Hilbert Transform
13. Properties of Fourier Transforms
14. Solved Problems
15. Assignment Questions
16. Quiz Questions

1. Fourier Series Representation of Continuous Time Periodic Signals:

Approximation or Representation of a continuous time periodic signal x(t) over a certain interval by a set of mutually orthogonal signals is called Fourier Series (FS). There are mainly two different ways of representing a continuous time periodic signal x(t) by a Fourier Series.

- Trigonometric Fourier Series
- Exponential Fourier Series

2. Dirichlet's Conditions:

Conditions under which a periodic signal x(t) can be represented by a Fourier Series is called existence of Fourier Series or Convergence of Fourier Series or Dirichlet conditions.

- **Condition-1:** Over any period T, the signal x(t) must be absolutely integrable;

$$\int_0^T |x(t)|dt < \infty$$

$$Example: x(t) = \frac{1}{t}, 0 < t \leq 1$$

Above periodic signal violates the first Dirichlet's condition.

- **Condition-2:** In any finite interval of time, the signal x(t) has only a finite number of maxima and minima.

$$Example: x(t) = Sin\left(\frac{2\pi}{t}\right), 0 < t \leq 1$$

Above periodic signal violates the second Dirichlet's condition but satisfies condition-1.

- **Condition-3:** In any finite interval of time, the signal x(t) has only a finite number of discontinuities. Furthermore, each of these discontinuities must be finite.

$$Example: x(t) = \begin{cases} 1; 0 \leq t < 4 \\ \frac{1}{2}; 4 \leq t < 6 \\ \frac{1}{4}; 6 \leq t < 7 \\ \frac{1}{8}; 7 \leq t < 7.5 \\ \quad\quad . \\ \quad\quad . \end{cases}$$

Above periodic signal violates the third Dirichlet's condition but satisfies conditions-1 & 2.

3. Trigonometric Fourier Series (TFS):

Approximation or Representation of a periodic signal x(t) over the interval (t_0, t_0+T) by a set of mutually orthogonal sinusoidal signals is called Trigonometric Fourier Series (TFS).

Now we can represent any periodic signal x(t) by using a set of mutually orthogonal sinusoidal signals, 1, Cos(w_0t), Cos(2w_0t), Cos(3w_0t),…….., Sin(w_0t), Sin(2w_0t), Sin(3w_0t),……

⇒ $x(t)=a_0 1+a_1 Cos(w_0t)+a_2Cos(2w_0t)+a_3Cos(3w_0t)+\dots+b_1Sin(w_0t)+b_2Sin(2w_0t)+b_3Sin(3w_0t)+\dots$

$$\Rightarrow x(t) = a_0 + \sum_{n=1}^{\infty}(a_n Cos(nw_0 t) + b_n Sin(nw_o t)),\ w_0 = \frac{2\pi}{T}$$

where, a_0, a_n and b_n are coefficients of TFS

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)dt$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)Cos(nw_0t)dt$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)Sin(nw_0t)dt$$

Proof:

$$a_n = \frac{\int_{t_0}^{t_0+T} x(t)Cos(nw_0t)dt}{\int_{t_0}^{t_0+T} Cos^2(nw_0t)dt}$$

$$= \frac{\int_{t_0}^{t_0+T} x(t)Cos(nw_0t)dt}{T/2}$$

$$= \frac{2}{T}\int_{t_0}^{t_0+T} x(t)Cos(nw_0t)dt$$

$$\int_{t_0}^{t_0+T} Cos^2(nw_0t)dt$$

$$= \int_{t_0}^{t_0+T}\left(\frac{1+Cos(2nw_0t)}{2}\right)dt$$

$$= \frac{1}{2}\left(t\Big|_{t_0}^{t_0+T} + \int_{t_0}^{t_0+T} Cos(2nw_0t)dt\right)$$

$$= \frac{1}{2}(t_0 + T - t_0 + 0)$$

$$= \frac{T}{2}$$

$$b_n = \frac{\int_{t_0}^{t_0+T} x(t) Sin(nw_0 t)dt}{\int_{t_0}^{t_0+T} Sin^2(nw_0 t)dt}$$

$$= \frac{\int_{t_0}^{t_0+T} x(t) Sin(nw_0 t)dt}{T/2}$$

$$= \frac{2}{T}\int_{t_0}^{t_0+T} x(t) Sin(nw_0 t)dt$$

$$\int_{t_0}^{t_0+T} Sin^2(nw_0 t)dt$$

$$= \int_{t_0}^{t_0+T} \left(\frac{1 - Cos(2nw_0 t)}{2}\right) dt$$

$$= \frac{1}{2}\left(t \Big|_{t_0}^{t_0+T} - \int_{t_0}^{t_0+T} Cos(2nw_0 t)dt \right)$$

$$= \frac{1}{2}(t_0 + T - t_0 - 0)$$

$$= \frac{T}{2}$$

$$a_0 = \frac{\int_{t_0}^{t_0+T} x(t) 1 dt}{\int_{t_0}^{t_0+T} 1^2 dt}$$

$$= \frac{\int_{t_0}^{t_0+T} x(t) dt}{t_0 + T - t_0}$$

$$= \frac{\int_{t_0}^{t_0+T} x(t) dt}{T}$$

$$= \frac{1}{T}\int_{t_0}^{t_0+T} x(t) dt$$

4. Exponential Fourier Series (EFS):

Approximation or Representation of a periodic signal x(t) over the interval (t_0, t_0+T) by a set of mutually orthogonal complex exponential signals is called Exponential Fourier Series (EFS).

Now represent a periodic signal x(t) by using a set of mutually orthogonal complex exponential signals … … … … e^{-j3w_0t}, e^{-j2w_0t}, e^{-jw_0t}, 1, e^{jw_0t}, e^{j2w_0t}, e^{j3w_0t}, … … … ….

$$\Rightarrow x(t) = \cdots C_{-3}e^{-j3w_0t} + C_{-2}e^{-j2w_0t} + C_{-1}e^{-jw_0t} + C_0 + C_1e^{jw_0t} + C_2e^{j2w_0t} + C_3e^{j3w_0t} + \cdots$$

$$\Rightarrow x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnw_0t}, \; w_0 = \frac{2\pi}{T}$$

where, C_n is the coefficient of EFS

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jnw_0t}dt$$

$$C_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)dt$$

Proof:

$$C_n = \frac{\int_{t_0}^{t_0+T} x(t)e^{-jnw_0t}dt}{\int_{t_0}^{t_0+T} e^{jnw_0t}e^{-jnw_0t}dt}$$

$$= \frac{\int_{t_0}^{t_0+T} x(t)e^{-jnw_0t}dt}{\int_{t_0}^{t_0+T} dt}$$

$$= \frac{\int_{t_0}^{t_0+T} x(t)e^{-jnw_0t}dt}{t_0 + T - t_0}$$

$$= \frac{\int_{t_0}^{t_0+T} x(t)e^{-jnw_0t}dt}{T}$$

$$= \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jnw_0t}dt$$

and

$$C_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)dt$$

5. Relation between TFS and EFS:

We know the Trigonometric Fourier Series (TFS) expansion of a periodic signal x(t);

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n Cos(nw_0 t) + b_n Sin(nw_o t)),\ w_0 = \frac{2\pi}{T}$$

where, a_0, a_n and b_n are coefficients of TFS

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)dt ------------- (1)$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)Cos(nw_0 t)dt ---------(2)$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)Sin(nw_0 t)dt ---------(3)$$

We know the Exponential Fourier Series (EFS) expansion of a periodic signal x(t) is

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnw_0 t},\ w_0 = \frac{2\pi}{T}$$

where, C_n is the coefficient of EFS

$$C_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)dt ------------(4)$$

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jnw_0 t}dt ---------(5)$$

Coefficients of TFS in terms EFS:

$a_0 = C_0$

$a_n = C_n + C_{-n}$

$b_n = j(C_n - C_{-n})$

Proof:

From equations (1) and (4)

$\Rightarrow a_0 = C_0$

From equation (2)

$$\Rightarrow a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)Cos(nw_0t)dt$$

$$\Rightarrow a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\left(\frac{e^{jnw_0t}+e^{-jnw_0t}}{2}\right)dt$$

$$\Rightarrow a_n = \frac{1}{T}\int_{t_0}^{t_0+T} \left(x(t)e^{jnw_0t} + x(t)e^{-jnw_0t}\right)dt$$

$$\Rightarrow a_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{jnw_0t}dt + \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jnw_0t}dt$$

$\Rightarrow a_n = C_{-n} + C_n = C_n + C_{-n}$

$\Rightarrow a_n = C_n + C_{-n}$

From equations (3)

$$\Rightarrow b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)Sin(nw_0t)dt$$

$$\Rightarrow b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\left(\frac{e^{jnw_0t}-e^{-jnw_0t}}{2j}\right)dt$$

$$\Rightarrow b_n = -j\frac{1}{T}\int_{t_0}^{t_0+T} \left(x(t)e^{jnw_0t} - x(t)e^{-jnw_0t}\right)dt$$

$$\Rightarrow b_n = -j\left(\frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{jnw_0t}dt - \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jnw_0t}dt\right)$$

$\Rightarrow b_n = -j(C_{-n} - C_n)$

$\Rightarrow b_n = j(C_n - C_{-n})$

Coefficients of EFS in terms TFS:

$$C_0 = a_0$$

$$C_n = \frac{a_n - jb_n}{2}$$

$$C_{-n} = \frac{a_n + jb_n}{2}$$

Proof:

From equations (1) and (4)

$$\Rightarrow C_0 = a_0$$

From equation (5)

$$\Rightarrow C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jnw_0t}dt$$

$$\Rightarrow C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\big(Cos(nw_0t) - jSin(nw_0t)\big)dt$$

$$\Rightarrow C_n = \frac{1}{T}\int_{t_0}^{t_0+T} \big(x(t)Cos(nw_0t) - jx(t)Sin(nw_0t)\big)dt$$

$$\Rightarrow C_n = \frac{1}{2}\left(\frac{2}{T}\int_{t_0}^{t_0+T} x(t)Cos(nw_0t)dt - j\frac{2}{T}\int_{t_0}^{t_0+T} x(t)Sin(nw_0t)dt\right)$$

$$\Rightarrow C_n = \frac{1}{2}(a_n - jb_n)$$

$$\Rightarrow C_n = \frac{a_n - jb_n}{2}$$

Similarly,

$$C_{-n} = \frac{a_n + jb_n}{2}$$

6. Properties of Fourier Series

We know the Exponential Fourier Series (EFS) expansion of a periodic signal x(t);

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnw_0 t},\ w_0 = \frac{2\pi}{T}$$

Where,

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jnw_0 t}dt$$

6.1. Parseval's Theorem

If x(t) is a periodic signal with exponential Fourier series coefficient C_n, then the average power of a signal x(t) can be computed from the formula

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |C_n|^2$$

Proof:

From the definition of average power of a signal

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2\, dt$$

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, x^*(t)\, dt$$

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\left(\sum_{n=-\infty}^{\infty} C_n e^{jnw_0 t}\right)^* dt$$

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\left(\sum_{n=-\infty}^{\infty} C_n^* e^{-jnw_0 t}\right) dt$$

$$P = \sum_{n=-\infty}^{\infty} C_n^* \left(\frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jnw_0 t}\, dt\right)$$

$$P = \sum_{n=-\infty}^{\infty} C_n^*\, C_n$$

$$P = \sum_{n=-\infty}^{\infty} |C_n|^2$$

6.2. Linear Property

If x(t) and y(t) are two periodic signals with exponential Fourier series coefficients C_n and D_n, then the exponential Fourier series coefficient of a signal ax(t)+by(t) is aC_n+bD_n.

Proof:

$$The\ coefficient\ of\ x(t), C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jnw_0t}dt$$

$$The\ coefficient\ of\ y(t), D_n = \frac{1}{T}\int_{t_0}^{t_0+T} y(t)e^{-jnw_0t}dt$$

$$The\ coefficient\ of\ ax(t) + by(t)$$

$$= \frac{1}{T}\int_{t_0}^{t_0+T} \big(ax(t) + by(t)\big)e^{-jnw_0t}dt$$

$$= \frac{1}{T}\int_{t_0}^{t_0+T} \big(ax(t)e^{-jnw_0t} + by(t)e^{-jnw_0t}\big)dt$$

$$= \frac{1}{T}\int_{t_0}^{t_0+T} ax(t)e^{-jnw_0t}dt + \frac{1}{T}\int_{t_0}^{t_0+T} by(t)e^{-jnw_0t}dt$$

$$= a\frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jnw_0t}dt + b\frac{1}{T}\int_{t_0}^{t_0+T} y(t)e^{-jnw_0t}dt$$

$$= aC_n + bD_n$$

6.3. Time Shifting Property

If x(t) is a periodic signal with exponential Fourier series coefficient C_n, then the exponential Fourier series coefficient of a signal $x(t - t_0)\ is\ C_n e^{-jnw_0t_0}$

Proof:

$$The\ coefficient\ of\ x(t), C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jnw_0t}dt$$

$$The\ coefficient\ of\ x(t - t_0)$$

$$= \frac{1}{T}\int_{t_0}^{t_0+T} x(t - t_0)e^{-jnw_0t}dt\, , let\ t - t_0 = \tau \Rightarrow dt = d\tau$$

$$= \frac{1}{T}\int_{0}^{T} x(\tau)e^{-jnw_0(t_0+\tau)}d\tau$$

$$= \frac{1}{T}\int_{0}^{T} x(\tau)e^{-jnw_0t_0}e^{-jnw_0\tau}d\tau$$

$$= e^{-jnw_0t_0}\frac{1}{T}\int_{0}^{T} x(\tau)e^{-jnw_0\tau}d\tau$$

$$= C_n e^{-jnw_0t_0}$$

6.4. Frequency Shifting Property

If x(t) is a periodic signal with exponential Fourier series coefficient C_n, then the exponential Fourier series coefficient of a signal $x(t)e^{-jmw_0t}$ $is\ C_{n-m}$

6.5. Time Reversal Property

If x(t) is a periodic signal with exponential Fourier series coefficient C_n, then the exponential Fourier series coefficient of a signal x(-t) is C_{-n}

6.6. Conjugate Property

If x(t) is a periodic signal with exponential Fourier series coefficient C_n, then the exponential Fourier series coefficient of a signal x*(t) is C_{-n}^*

6.7. Time Scaling Property

If x(t) is a periodic signal with exponential Fourier series coefficient C_n, then the exponential Fourier series coefficient of a signal x(at) is C_n, where, a > 0.

6.8. Time Differentiation Property

If x(t) is a periodic signal with exponential Fourier series coefficient C_n, then the exponential Fourier series coefficient of a signal $\frac{d}{dt}x(t)\ is\ jnw_0C_n$

6.9. Time Integration Property

If x(t) is a periodic signal with exponential Fourier series coefficient C_n, then the exponential Fourier series coefficient of a signal $\int_{-\infty}^{t} x(\tau)d\tau\ \ is\ \frac{C_n}{jnw_0}$

6.10. Periodic Convolution

If x(t) and y(t) are two periodic signals with equal period of T and the exponential Fourier series coefficients are C_n and D_n, then the exponential Fourier series coefficient of a convoluted signal $x(t) * y(t)\ \ is\ \ TC_nD_n$

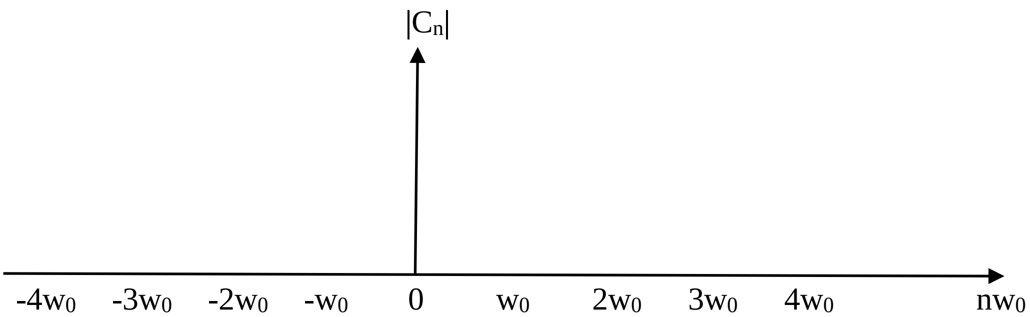
6.11. Multiplication Theorem

If x(t) and y(t) are two periodic signals with equal period of T and the exponential Fourier series coefficients are C_n and D_n, then the exponential Fourier series coefficient of a signal

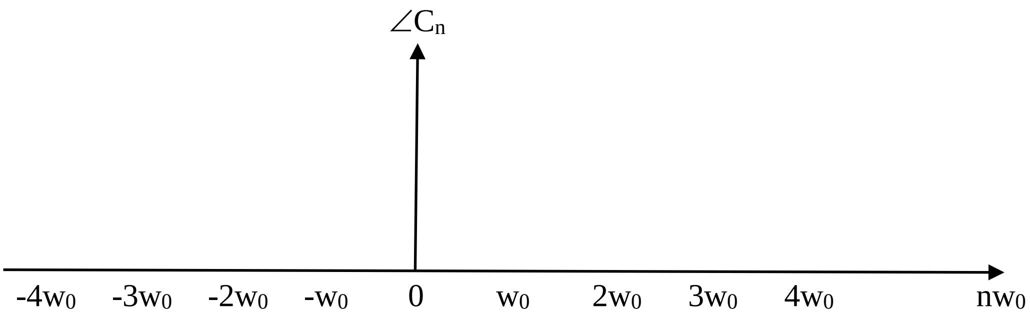
$$x(t)y(t)\ \ is\ \sum_{m=-\infty}^{\infty} C_m D_{n-m}$$

7. Complex Fourier Spectrum:

- The coefficient of EFS is complex, i.e. $C_n = |C_n|\angle C_n$.
- Graphical representation of $|C_n|$ Vs $w=nw_0$ is called Line Spectrum or Magnitude Spectrum.



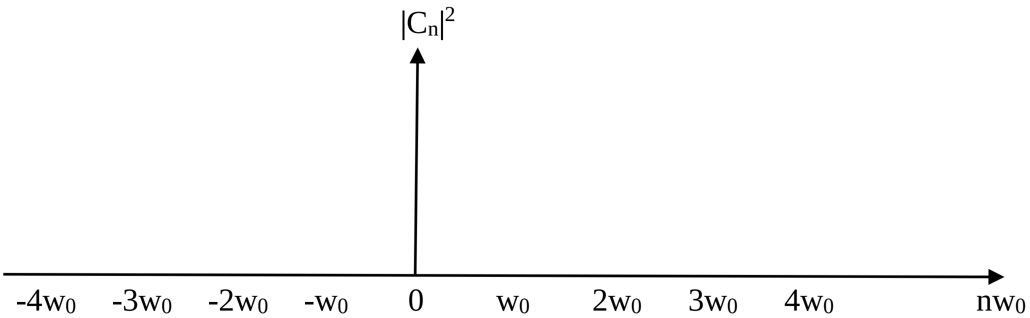
- Graphical representation of $\angle C_n$ Vs $w=nw_0$ is called Phase Spectrum.



- Graphical representation of $|C_n|$ Vs $w=nw_0$ & $\angle C_n$ Vs $w=nw_0$ is called Complex Fourier Spectrum.
- Amplitude Spectrum exhibits even symmetry.
- Phase Spectrum exhibits odd symmetry.
- When x(t) is real , then $C_{-n}=C^*_n$. i.e. C_n and C_{-n} are complex conjugate pairs.
- Average power of a periodic signal x(t) can be computed from parseval's theorem

$$P = \sum_{n=-\infty}^{\infty} |C_n|^2$$

- Graphical representation of $|C_n|^2$ Vs $w=nw_0$ is called Power Spectrum.



8. Deriving Fourier Transform from Fourier Series:

We know that any periodic signal can be represented as the linear combination of complex exponential signals and such a representation is called Complex Exponential Fourier Series or Exponential Fourier Series or Fourier Series.

Fourier Series representation of a periodic signal x(t) over the interval (t_0, t_0+T) is

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnw_0 t}$$

Where,

$$w_0 = \frac{2\pi}{T}$$

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jnw_0 t}dt$$

Let, nw_0 = w and t_0=-T/2

$$\Rightarrow TC_n = \int_{-T/2}^{T/2} x(t)e^{-jwt}dt$$

apply as limit T→∞

$$\underset{T\to\infty}{\text{Lt}}(\text{T}c_n) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

Result of above integration is the function of frequency 'w' and it is represented with X(w) or X(f)

$$\Rightarrow X(w) = FT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

It is the Fourier Transform of a signal x(t).

Fourier Transform is a mathematical tool, which is used to obtain the frequency domain representation of a given continuous time domain aperiodic signal.

9. Fourier Transform of Arbitrary Signals:

Fourier Transform of an arbitrary signal x(t) can be computed from the formula

$$FT[x(t)] = X(jw) = X(w) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

or

$$FT[x(t)] = X(j2\pi f) = X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

Evaluation of time domain signal x(t) from the frequency domain is called Inverse Fourier Transform

$$IFT[X(w)] = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)e^{jwt}dw = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

Where, the frequency 'w' is measured in rad/sec and 'f' is measured in Hz.

x(t) and X(w) are Fourier Transformable pairs

$$x(t) \underset{IFT}{\overset{FT}{\rightleftarrows}} X(w)$$

In general, the frequency domain X(w) is in complex form and it can be expressed as

$$X(w) = X_R(w) + j\,X_I(w)$$

Where,

$X_R(w)$: Real part of $X(w)$

$X_I(w)$: Imaginary part of $X(w)$

Magnitude of X(w) is called the magnitude spectrum and it can be computed from the formula

$$|X(w)| = \sqrt{[X_R(w)]^2 + [X_I(w)]^2}$$

Phase of X(w) is called the phase spectrum and it can be computed from the formula

$$\angle X(w) = Tan^{-1}\left(\frac{X_I(w)}{X_R(w)}\right)$$

10. Fourier Transform of Standard Signals:

Fourier Transform of Standard Signals, like impulse signal, exponential signals, rectangular signal, triangular signal, DC signal, complex exponential signal, sinusoidal signal, step signal, signum signal, etc., can be computed.

(1) $FT[\delta(t)] = 1$

(2) $FT[e^{-at}u(t)] = \frac{1}{a+jw}$

(3) $FT[te^{-at}u(t)] = \frac{1}{(a+jw)^2}$

(4) $FT[t^n e^{-at}u(t)] = \frac{n!}{(a+jw)^{n+1}}$

(5) $FT[e^{at}u(-t)] = \frac{1}{a-jw}$

(6) $FT\left[e^{-a|t|}\right] = FT[e^{at}u(-t) + e^{-at}u(t)] = \frac{2a}{a^2+w^2}$

(7) $FT\left[Arect\left(\frac{t}{T}\right)\right] = ATSinc(fT) = ATSa\left(\frac{wT}{2}\right)$

(8) $FT\left[A.tri\left(\frac{t}{T}\right)\right] = \frac{AT}{2}Sinc^2\left(\frac{fT}{2}\right) = \frac{AT}{2}Sa^2\left(\frac{wT}{4}\right)$

(9) $FT[1] = 2\pi\delta(w) = \delta(f)$

(10) $FT[e^{jw_0t}] = 2\pi\delta(w-w_0)$

(11) $FT[Cos(w_0t)] = \pi(\delta(w+w_0)+\delta(w-w_0))$

(12) $FT[Sin(w_0t)] = j\pi(\delta(w+w_0)-\delta(w-w_0))$

(13) $FT[Sgn(t)] = FT\left[\underset{a\to 0}{Lt} e^{-a|t|}Sgn(t)\right] = \frac{2}{jw} = \frac{1}{j\pi f}$

(14) $FT[u(t)] = FT\left[\frac{1+Sgn(t)}{2}\right] = \pi\delta(w) + \frac{1}{jw}$

(15) $FT[Cos(w_0t)u(t)] = \frac{\pi}{2}(\delta(w+w_0)+\delta(w-w_0)) + \frac{jw}{{w_0}^2-w^2}$

(16) $FT[Sin(w_0t)u(t)] = j\frac{\pi}{2}(\delta(w+w_0)-\delta(w-w_0)) + \frac{w}{{w_0}^2-w^2}$

(17) $FT\left[e^{-kt^2}\right] = \sqrt{\frac{\pi}{k}}e^{-\frac{w^2}{4k}} = \sqrt{\frac{\pi}{k}}e^{-\frac{\pi^2f^2}{k}}$

(18) $FT\left[e^{-\pi t^2}\right] = e^{-\pi f^2}$

(19) $FT\sum_{n=-\infty}^{\infty}\delta(t-nTs) = w_s\sum_{n=-\infty}^{\infty}\delta(w-nw_s) = f_s\sum_{n=-\infty}^{\infty}\delta(f-nf_s) = \frac{1}{Ts}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{Ts}\right)$

11. Fourier Transform of Periodic Signals:

We know the Fourier Series representation of periodic signal

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnw_0 t}$$

Apply Fourier Transform

$$FT[x(t)] = \sum_{n=-\infty}^{\infty} C_n FT[e^{jnw_0 t}]\,; use\ FT[e^{jw_0 t}] = 2\pi\delta(w - w_0)$$

$$= \sum_{n=-\infty}^{\infty} C_n 2\pi\delta(w - nw_0)$$

$$= 2\pi \sum_{n=-\infty}^{\infty} C_n \delta(w - nw_0)$$

Where,

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jnw_0 t} dt$$

12. Introduction to Hilbert Transform:

- Convolution of a signal x(t) with $'\frac{1}{\pi t}'$ is called Hilbert Transform of a signal x(t).

$$HT[x(t)] = \hat{x}(t) = x(t) * \frac{1}{\pi t} = \int_{-\infty}^{\infty} x(\tau)\frac{1}{\pi(t-\tau)} d\tau = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} d\tau$$

- In the case of Hilbert Transformation of a signal, the magnitude spectrum of the signal does not change, only phase spectrum of the signal is changed.
- Hilbert Transform of a signal does not change the domain of the signal.
- Hilbert Transform of the signal x(t) is a linear operation.
- Hilbert Transform is applicable to all the signals which are Fourier transformable.
- The process of recovering the original signal x(t) from its Hilbert Transform is called the Inverse Hilbert Transform. Mathematically, it is defined as

$$IHT[\hat{x}(t)] = x(t) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\hat{x}(\tau)}{t-\tau} d\tau$$

- The signal x(t) and its Hilbert Transform $\hat{x}(t)$ are called Hilbert Transform pair.
- Example: HT[Sin(w_0t)] = - Cos(w_0t)

13. Properties of Fourier Transform:

Various properties used in Fourier Transform are Linearity, Time Shifting, Frequency Shifting, Time Reversal, Conjugation or Conjugate Symmetry, Time or Frequency Scaling, Time Differentiation, Time Integration, Frequency Differentiation, Frequency Integration, Duality, Parseval's Relation for Aperiodic Signals, Time Convolution Theorem or Convolution Property and Frequency Convolution Theorem or Multiplication Property.

13.1. Linear Property:

If $x_1(t)$, $x_2(t)$ are two continuous time aperiodic signals and $FT[x_1(t)] = X_1(w)$, $FT[x_2(t)] = X_2(w)$, then $FT[a\,x_1(t) + b\,x_2(t)] = a\,X_1(w) + b\,X_2(w)$ is called linear property of Fourier Transform.

Proof:

From the definition of Fourier Transform

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

Replace x(t) with **a x_1(t) + b x_2(t)**

$$FT[ax_1(t) + bx_2(t)] = \int_{-\infty}^{\infty} \left(ax_1(t) + bx_2(t)\right)e^{-jwt}dt$$

$$= \int_{-\infty}^{\infty} \left(ax_1(t)e^{-jwt} + bx_2(t)e^{-jwt}\right)dt$$

$$= \int_{-\infty}^{\infty} ax_1(t)e^{-jwt}dt + \int_{-\infty}^{\infty} bx_2(t)e^{-jwt}dt$$

$$= a\int_{-\infty}^{\infty} x_1(t)e^{-jwt}dt + b\int_{-\infty}^{\infty} x_2(t)e^{-jwt}dt$$

$$= aFT[x_1(t)] + bFT[x_2(t)]$$

$$= a\,X_1(w) + b\,X_2(w)$$

13.2. Time Shifting Property:

If x(t) is a continuous time aperiodic signal and FT[x(t)] = X(w), then FT[x(t – t_0)] = $e^{-j\omega t_0}$ X(w) is called time shifting property of Fourier Transform.

Proof:

From the definition of Fourier Transform

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

Replace x(t) with x(t – t_0)

$$FT[x(t-t_0)] = \int_{-\infty}^{\infty} x(t-t_0)e^{-jwt}dt, Let\ t-t_0=\tau, dt=d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-jw(t_0+\tau)}d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-jwt_0}e^{-jw\tau}d\tau$$

$$= e^{-jwt_0}\int_{-\infty}^{\infty} x(\tau)e^{-jw\tau}d\tau$$

$$= e^{-jwt_0}\ FT[x(t)]$$

$$= e^{-jwt_0}\ X(w)$$

13.3. Frequency Shifting Property:

If x(t) is continuous time aperiodic signal and FT[x(t)] = X(w), then FT[e^{jwot} x(t)] = X(w-w_o) is called frequency shifting property of Fourier Transform.

Proof:

From the definition of Fourier Transform

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

Replace x(t) with e^{jwot} x(t)

$$FT[e^{jw_0t}x(t)] = \int_{-\infty}^{\infty} e^{jw_0t}x(t)e^{-jwt}dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-j(w-w_0)t}dt\ ;\ FT[x(t)] = X(w) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$= X(w-w_0)$$

13.4. Time Reversal Property:

If x(t) is a continuous time aperiodic signal and FT[x(t)] = X(w), then FT[x(–t)] = X(-w) is called time reversal property of Fourier Transform.

Proof:

From the definition of Fourier Transform

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

Replace x(t) with x(-t)

$$FT[x(-t)] = \int_{-\infty}^{\infty} x(-t)e^{-jwt}dt\,, Let -t = \tau \Rightarrow dt = -d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-jw(-\tau)}d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-j(-w)\tau}d\tau\,; FT[x(t)] = X(w) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$= X(-w)$$

13.5. Conjugation or Conjugate Property:

If x(t) is a continuous time aperiodic signal and FT[x(t)] = X(w), then FT[x*(t)] = X*(-w) is called conjugate property of Fourier Transform.

Proof:

From the definition of Fourier Transform

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

Replace x(t) with x*(t)

$$FT[x^*(t)] = \int_{-\infty}^{\infty} x^*(t)e^{-jwt}dt$$

$$= \left(\int_{-\infty}^{\infty} x(t)e^{jwt}dt\right)^*$$

$$= \left(\int_{-\infty}^{\infty} x(t)e^{-j(-w)t}dt\right)^*$$

$$= (X(-w))^*$$

$$= X^*(-w)$$

13.6. Time or Frequency Scaling Property:

If x(t) is a continuous time aperiodic signal and FT[x(t)] = X(w), then $FT[x(at)] = \frac{1}{|a|}X\left(\frac{w}{a}\right)$ is called time and frequency scaling property of Fourier Transform.

Proof:

From the definition of Fourier Transform

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

Case-1 (a > 0) : Replace x(t) with x(at)

$$FT[x(at)] = \int_{-\infty}^{\infty} x(at)e^{-jwt}dt\,, Let\ at = \tau \Rightarrow adt = d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-jw\tau/a}(d\tau/a)$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)e^{-j(\frac{w}{a})\tau}d\tau;\ FT[x(t)] = X(w) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$= \frac{1}{a}X\left(\frac{w}{a}\right) - - - - - - -(1)$$

Case-2 (a > 0): Replace x(t) with x(-at)

$$FT[x(-at)] = \int_{-\infty}^{\infty} x(-at)e^{-jwt}dt\,, Let - at = \tau \Rightarrow adt = -d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-jw\left(-\frac{\tau}{a}\right)}(d\tau/a)$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)e^{-j(\frac{w}{-a})\tau}d\tau$$

$$= \frac{1}{a}X\left(\frac{w}{-a}\right) - - - - - (2)$$

Compare equations (1) and (2)

$$\Rightarrow FT[x(at)] = \frac{1}{|a|}X\left(\frac{w}{a}\right)$$

Note: If the time domain signal x(t) is scaled with ‘a’ then the frequency domain X(w) is scaled with ‘1/a’. Hence the property is called time scaling or frequency scaling property.

13.7. Convolution (Time Convolution) Theorem:

If $x_1(t)$, $x_2(t)$ are two continuous time aperiodic signals and $FT[x_1(t)] = X_1(w)$, $FT[x_2(t)] = X_2(w)$, then $FT[x_1(t) * x_2(t)] = X_1(w) X_2(w)$ is called time convolution theorem of Fourier Transform. i.e, convolution in time domain leads to multiplication in frequency domain.

Proof:

From the definition of Fourier Transform

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

Replace x(t) with $x_1(t) * x_2(t)$

$$FT[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} (x_1(t) * x_2(t))e^{-jwt}dt\,;\; x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)d\tau\right) e^{-jwt}dt$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \left(\int_{-\infty}^{\infty} x_2(t-\tau)e^{-jwt}\, dt\right) d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau) FT[x_2(t-\tau)]d\tau\,;\; FT[x(t-t_0)] = e^{-jwt_0}\, X(w)$$

$$= \int_{-\infty}^{\infty} x_1(\tau)e^{-jw\tau}X_2(w)d\tau$$

$$= X_2(w)\int_{-\infty}^{\infty} x_1(\tau)e^{-jw\tau}d\tau$$

$$= X_2(w)FT[x_1(t)]$$

$$= X_1(w)X_2(w)$$

13.8. Multiplication or Frequency Convolution Theorem:

If $x_1(t)$, $x_2(t)$ are two continuous time aperiodic signals and $FT[x_1(t)] = X_1(w)$, $FT[x_2(t)] = X_2(w)$, then $FT[x_1(t)x_2(t)] = \frac{X_1(w)*X_2(w)}{2\pi}$ is called frequency convolution theorem of Fourier Transform. i.e, convolution in frequency domain leads to multiplication in time domain.

13.9. Time Differentiation Property:

If x(t) is a continuous time aperiodic signal and FT[x(t)] = X(w), then $FT\left[\frac{d}{dt}x(t)\right] = jwX(w)$ is called time differentiation property of Fourier Transform.

Proof:

From the definition of Inverse Fourier Transform

$$IFT[X(w)] = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)e^{jwt}dw$$

Differentiate w.r.t 't'

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)\frac{d}{dt}\left(e^{jwt}\right)dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)\left(jwe^{jwt}\right)dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} jwX(w)e^{jwt}dw$$

$$= IFT[jwX(w)]$$

$$\Rightarrow FT\left[\frac{d}{dt}x(t)\right] = jwX(w)$$

13.10. Frequency Differentiation Property:

If x(t) is a continuous time aperiodic signal and FT[x(t)] = X(w), then $FT[tx(t)] = j\frac{d}{dw}X(w)$ is called time differentiation property of Fourier Transform.

Proof:

From the definition of Fourier Transform

$$FT[x(t)] = X(w) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

Differentiate X(w) w.r.t 'w'

$$\frac{d}{dw}X(w) = \int_{-\infty}^{\infty} x(t)\frac{d}{dw}\left(e^{-jwt}\right)dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-jwt}(-jt)dt$$

$$= -j\int_{-\infty}^{\infty} tx(t)e^{-jwt}dt$$

$$= -jFT[tx(t)]$$

$$\Rightarrow FT[tx(t)] = j\frac{d}{dw}X(w)$$

13.11. Time Integration Property:

If x(t) is a continuous time aperiodic signal and FT[x(t)] = X(w), then $FT[\int x(t)dt] = \frac{X(w)}{jw}$ is called time integration property of Fourier Transform.

Proof: From the definition of Inverse Fourier Transform

$$IFT[X(w)] = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)e^{jwt}dw$$

Integrate w.r.t 't'

$$\int x(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)\left(\int e^{jwt}dt\right)dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)\left(\frac{e^{jwt}}{jw}\right)dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{X(w)}{jw}e^{jwt}dw$$

$$= IFT\left[\frac{X(w)}{jw}\right]$$

$$\Rightarrow FT\left[\int x(t)dt\right] = \frac{X(w)}{jw}$$

13.12. Frequency Integration Property:

If x(t) is a continuous time aperiodic signal and FT[x(t)] = X(w), then $FT\left[\frac{x(t)}{t}\right] = -j\int X(w)dw$ is called frequency differentiation property of Fourier Transform.

Proof: From the definition of Fourier Transform

$$FT[x(t)] = X(w) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

Integrate X(w) w.r.t 'w'

$$\int X(w)dw = \int_{-\infty}^{\infty} x(t)\left(\int e^{-jwt}dw\right)dt$$

$$= \int_{-\infty}^{\infty} x(t)\left(\frac{e^{-jwt}}{-jt}\right)dt$$

$$= -\frac{1}{j}\int_{-\infty}^{\infty} \frac{x(t)}{t}e^{-jwt}dt$$

$$= -\frac{1}{j}FT\left[\frac{x(t)}{t}\right]$$

$$\Rightarrow FT\left[\frac{x(t)}{t}\right] = -j\int X(w)dw$$

13.13. Duality Property:

If x(t) is a continuous time aperiodic signal and FT[x(t)] = X(w),

then $FT[X(t)] = 2\pi x(-w) = x(-f)$ is called duality property of Fourier Transform.

Proof:

From the definition of Inverse Fourier Transform

$$IFT[X(w)] = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)e^{jwt}dw$$

$$\Rightarrow 2\pi x(t) = \int_{-\infty}^{\infty} X(w)e^{jwt}dw, repalce\ 't'\ with\ '-t'$$

$$\Rightarrow 2\pi x(-t) = \int_{-\infty}^{\infty} X(w)e^{-jwt}dw, repalce\ 'w'\ with\ '\tau'$$

$$\Rightarrow 2\pi x(-t) = \int_{-\infty}^{\infty} X(\tau)e^{-j\tau t}d\tau, repalce\ 't'\ with\ 'w'$$

$$\Rightarrow 2\pi x(-w) = \int_{-\infty}^{\infty} X(\tau)e^{-j\tau w}d\tau, repalce\ '\tau'\ with\ 't'$$

$$\Rightarrow 2\pi x(-w) = \int_{-\infty}^{\infty} X(t)e^{-jwt}dt$$

$$\Rightarrow 2\pi x(-w) = FT[X(t)]$$

$$\Rightarrow FT[X(t)] = 2\pi x(-w)$$

Similarly, we can show that

$$\Rightarrow FT[X(t)] = x(-f)$$

13.14. Parseval's Theorem:

If x(t) is a continuous time aperiodic signal and FT[x(t)] = X(w), then the total energy under the signal x(t) can be computed from x(t) as well as X(w) through the Parsevalls theorem

$$E = \int_{-\infty}^{\infty} |\, x(t) \,|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\, X(w) \,|^2 \, dw$$

Proof:

We know that the total Energy under the signal x(t) can be computed from the formula

$$E = \int_{-\infty}^{\infty} |\, x(t) \,|^2 dt$$

$$= \int_{-\infty}^{\infty} x(t)x^*(t)dt \,;\, IFT[X(w)] = x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(w)e^{jwt} dw$$

$$= \int_{-\infty}^{\infty} x(t) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(w)e^{jwt} dw \right)^* dt$$

$$= \int_{-\infty}^{\infty} x(t) \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(w)e^{-jwt} dw \right) dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(w) \left(\int_{-\infty}^{\infty} x(t)e^{-jwt} dt \right) dw$$

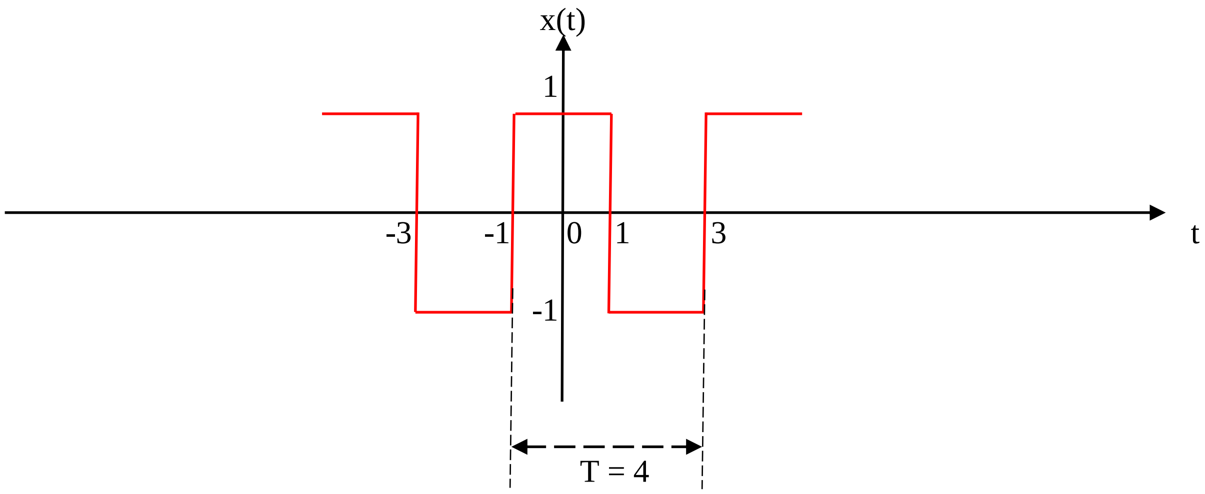
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(w) FT[x(t)] dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(w)X(w) dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\, X(w) \,|^2 \, dw$$

14. Solved Problem:

(14.1) Determine the Trigonometric Fourier Series expansion of a periodic signal x(t) as shown



Given periodic signal x(t) over one period (-1, 3)

$$x(t) = \begin{cases} 1, & -1 < t < 1 \\ -1, & 1 < t < 3 \end{cases}$$

Trigonometric Fourier Series (TFS) expansion of a periodic signal x(t) is

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n Cos(nw_0 t) + b_n Sin(nw_o t)\right),\ w_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n Cos\left(\frac{n\pi}{2}t\right) + b_n Sin\left(\frac{n\pi}{2}t\right)\right)$$

where, a_0, a_n and b_n are coefficients of TFS

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)dt$$

$$= \frac{1}{4}\int_{-1}^{3} x(t)dt$$

$$= \frac{1}{4}\left(\int_{-1}^{1} x(t)dt + \int_{1}^{3} x(t)dt\right)$$

$$= \frac{1}{4}\left(\int_{-1}^{1} (1)dt + \int_{1}^{3} (-1)dt\right)$$

$$= \frac{1}{4}\left((t)\Big|_{-1}^{1} + (-t)\Big|_{1}^{3}\right)$$

$$= \frac{1}{4}((1+1) + (-3+1))$$

$$= \frac{1}{4}(2-2) = 0$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)Cos(nw_0t)dt$$

$$= \frac{2}{4}\int_{-1}^{3} x(t)Cos\left(\frac{n\pi}{2}t\right)dt$$

$$= \frac{1}{2}\left(\int_{-1}^{1}(1)Cos\left(\frac{n\pi}{2}t\right)dt + \int_{1}^{3}(-1)Cos\left(\frac{n\pi}{2}t\right)dt\right)$$

$$= \frac{1}{2}\left(\frac{Sin\left(\frac{n\pi}{2}t\right)}{\frac{n\pi}{2}}\Bigg|_{-1}^{1} - \frac{Sin\left(\frac{n\pi}{2}t\right)}{\frac{n\pi}{2}}\Bigg|_{1}^{3}\right)$$

$$= \frac{1}{2}\left(\frac{Sin\left(\frac{n\pi}{2}\right) - Sin\left(-\frac{n\pi}{2}\right)}{\frac{n\pi}{2}} - \frac{Sin\left(\frac{3n\pi}{2}\right) - Sin\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{2}}\right)$$

$$= \frac{1}{n\pi}\left(Sin\left(\frac{n\pi}{2}\right) + Sin\left(\frac{n\pi}{2}\right) - Sin\left(\frac{3n\pi}{2}\right) + Sin\left(\frac{n\pi}{2}\right)\right)$$

$$= \frac{1}{n\pi}\left(3Sin\left(\frac{n\pi}{2}\right) - Sin\left(\frac{3n\pi}{2}\right)\right)$$

$$= \frac{1}{n\pi}\left(3Sin\left(\frac{n\pi}{2}\right) - \left(3Sin\left(\frac{n\pi}{2}\right) - 4Sin^3\left(\frac{n\pi}{2}\right)\right)\right)$$

$$= \frac{4}{n\pi}Sin^3\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow a_1 = \frac{4}{\pi}Sin^3\left(\frac{\pi}{2}\right) = \frac{4}{\pi}(1)^3 = \frac{4}{\pi}$$

$$\Rightarrow a_2 = \frac{4}{2\pi}Sin^3\left(\frac{2\pi}{2}\right) = \frac{2}{\pi}Sin^3(\pi) = \frac{2}{\pi}(0)^3 = 0$$

$$\Rightarrow a_3 = \frac{4}{3\pi}Sin^3\left(\frac{3\pi}{2}\right) = \frac{4}{3\pi}(-1)^3 = -\frac{4}{3\pi}$$

$$\Rightarrow a_4 = \frac{4}{4\pi}Sin^3\left(\frac{4\pi}{2}\right) = \frac{1}{\pi}Sin^3(2\pi) = \frac{1}{\pi}(0)^3 = 0$$

$$\Rightarrow a_5 = \frac{4}{5\pi}Sin^3\left(\frac{5\pi}{2}\right) = \frac{4}{5\pi}(1)^3 = \frac{4}{5\pi}$$

Similarly,

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)Sin(nw_0t)dt$$

$$= \frac{2}{4}\int_{-1}^{3} x(t)Sin\left(\frac{n\pi}{2}t\right)dt$$

$$= \frac{1}{2}\left(\int_{-1}^{1}(1)Sin\left(\frac{n\pi}{2}t\right)dt + \int_{1}^{3}(-1)Sin\left(\frac{n\pi}{2}t\right)dt\right)$$

$$= \frac{1}{2}\left(\frac{-Cos\left(\frac{n\pi}{2}t\right)}{\frac{n\pi}{2}} \Bigg|_{-1}^{1} - \frac{-Cos\left(\frac{n\pi}{2}t\right)}{\frac{n\pi}{2}} \Bigg|_{1}^{3} \right)$$

$$= \frac{1}{2}\left(\frac{-Cos\left(\frac{n\pi}{2}\right) + Cos\left(-\frac{n\pi}{2}\right)}{\frac{n\pi}{2}} + \frac{Cos\left(\frac{3n\pi}{2}\right) - Cos\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{2}} \right)$$

$$= \frac{1}{n\pi}\left(-Cos\left(\frac{n\pi}{2}\right) + Cos\left(\frac{n\pi}{2}\right) + Cos\left(\frac{3n\pi}{2}\right) - Cos\left(\frac{n\pi}{2}\right) \right)$$

$$= \frac{1}{n\pi}\left(Cos\left(\frac{3n\pi}{2}\right) - Cos\left(\frac{n\pi}{2}\right) \right)$$

$$= \frac{1}{n\pi}\left(4Cos^3\left(\frac{n\pi}{2}\right) - 3Cos\left(\frac{n\pi}{2}\right) - Cos\left(\frac{n\pi}{2}\right) \right)$$

$$= \frac{1}{n\pi}\left(4Cos^3\left(\frac{n\pi}{2}\right) - 4Cos\left(\frac{n\pi}{2}\right) \right)$$

$$= \frac{4}{n\pi}\left(Cos^3\left(\frac{n\pi}{2}\right) - Cos\left(\frac{n\pi}{2}\right) \right)$$

$$\Rightarrow b_1 = \frac{4}{\pi}\left(Cos^3\left(\frac{\pi}{2}\right) - Cos\left(\frac{\pi}{2}\right) \right) = \frac{4}{\pi}(0 - 0) = 0$$

$$\Rightarrow b_2 = \frac{4}{2\pi}\left(Cos^3\left(\frac{2\pi}{2}\right) - Cos\left(\frac{2\pi}{2}\right) \right) = \frac{2}{\pi}\left(Cos^3(\pi) - Cos(\pi)\right) = \frac{2}{\pi}(-1 + 1) = 0$$

$$\Rightarrow b_3 = \frac{4}{3\pi}\left(Cos^3\left(\frac{3\pi}{2}\right) - Cos\left(\frac{3\pi}{2}\right) \right) = \frac{4}{3\pi}(0 - 0) = 0$$

$$\Rightarrow b_4 = \frac{4}{4\pi}\left(Cos^3\left(\frac{4\pi}{2}\right) - Cos\left(\frac{4\pi}{2}\right) \right) = \frac{1}{\pi}\left(Cos^3(2\pi) - Cos(2\pi)\right) = \frac{1}{\pi}(1 - 1) = 0$$

$$\Rightarrow b_n = 0, for\ all\ values\ of\ n$$

We know the Trigonometric Fourier Series (TFS) expansion of a periodic signal x(t) is

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n Cos\left(\frac{n\pi}{2}t\right) + b_n Sin\left(\frac{n\pi}{2}t\right) \right)$$

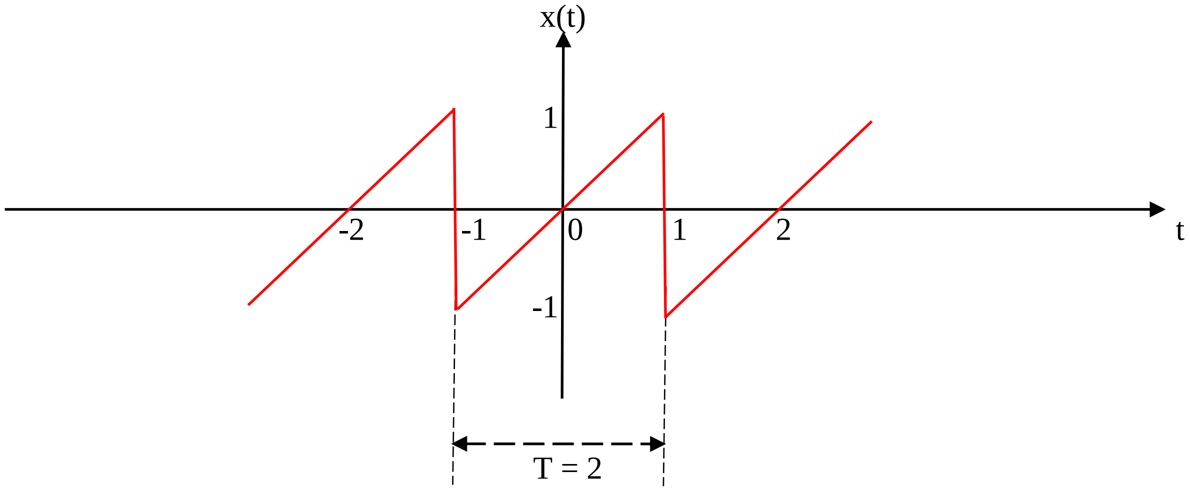
Now substitute a_0, a_n and b_n in x(t)

$$x(t) = a_1 Cos\left(\frac{\pi}{2}t\right) + a_2 Cos\left(\frac{2\pi}{2}t\right) + a_3 Cos\left(\frac{3\pi}{2}t\right) + a_4 Cos\left(\frac{4\pi}{2}t\right) + a_5 Cos\left(\frac{5\pi}{2}t\right) + \cdots$$

$$x(t) = \frac{4}{\pi}Cos\left(\frac{\pi}{2}t\right) + 0Cos\left(\frac{2\pi}{2}t\right) - \frac{4}{3\pi}Cos\left(\frac{3\pi}{2}t\right) + 0Cos\left(\frac{4\pi}{2}t\right) + \frac{4}{5\pi}Cos\left(\frac{5\pi}{2}t\right) + \cdots$$

$$x(t) = \frac{4}{\pi}\left(Cos\left(\frac{\pi}{2}t\right) - \frac{1}{3}Cos\left(\frac{3\pi}{2}t\right) + \frac{1}{5}Cos\left(\frac{5\pi}{2}t\right) - \frac{1}{7}Cos\left(\frac{7\pi}{2}t\right) + \cdots \right)$$

(14.2) Determine the Trigonometric Fourier Series expansion of a periodic signal x(t) as shown



Given periodic signal x(t) over one period (-1, 1)

$x(t) = t, -1 < t < 1$

Trigonometric Fourier Series (TFS) expansion of a periodic signal x(t) is

$$x(t) = a_0 + \sum_{n=1}^{\infty}\big(a_n Cos(nw_0 t) + b_n Sin(nw_o t)\big),\ w_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$$

$$x(t) = a_0 + \sum_{n=1}^{\infty}\big(a_n Cos(n\pi t) + b_n Sin(n\pi t)\big)$$

where, a_0, a_n and b_n are coefficients of TFS

$$a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)dt$$

$$= \frac{1}{2}\int_{-1}^{1} t\ dt$$

$$= \frac{1}{2}(0), because \int_{-a}^{a} x(t)dt = 0, if\ x(t) is\ odd$$

$$= 0$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)Cos(nw_0 t)dt$$

$$= \frac{2}{2}\int_{-1}^{1} t\ Cos(n\pi t)dt$$

$$= \int_{-1}^{1} t\ Cos(n\pi t)dt\ , \int_{-a}^{a} x(t)dt = 0, if\ x(t) is\ odd$$

$$= 0$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)Sin(nw_0 t)dt$$

$$= \frac{2}{2}\int_{-1}^{1} t\,Sin(n\pi t)dt$$

$$= \int_{-1}^{1} t\,Sin(n\pi t)dt\,, \int_{-a}^{a} x(t)dt = 2\int_{0}^{a} x(t)dt, if\ x(t) is\ even$$

$$= 2\int_{0}^{1} t\,Sin(n\pi t)dt$$

$$= 2\left(t\frac{-Cos(n\pi t)}{n\pi} - \int_{0}^{1}(1)\frac{-Cos(n\pi t)}{n\pi}dt\right)$$

$$= 2\left(\frac{-tCos(n\pi t)}{n\pi} + \frac{Sin(n\pi t)}{(n\pi)^2}\right)\Bigg|_0^1$$

$$= 2\left(\frac{-Cos(n\pi) + 0}{n\pi} + \frac{Sin(n\pi) - 0}{(n\pi)^2}\right)$$

$$= 2\left(\frac{-(-1)^n}{n\pi} + \frac{0}{(n\pi)^2}\right)$$

$$= -\frac{2}{n\pi}(-1)^n$$

$$\Rightarrow b_1 = -\frac{2}{\pi}(-1) = \frac{2}{\pi}$$

$$\Rightarrow b_2 = -\frac{2}{2\pi}(1) = -\frac{2}{2\pi}$$

$$\Rightarrow b_3 = -\frac{2}{3\pi}(-1) = \frac{2}{3\pi}$$

$$\Rightarrow b_4 = -\frac{2}{4\pi}(1) = -\frac{2}{4\pi}$$

We know that the Trigonometric Fourier Series (TFS) expansion of a periodic signal x(t) is

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n Cos(n\pi t) + b_n Sin(n\pi t)\right)$$

Now substitute a_0, a_n and b_n in x(t)

$$x(t) = b_1 Sin(\pi t) + b_2 Sin(2\pi t) + b_3 Sin(3\pi t) + b_4 Sin(4\pi t) + b_5 Sin(5\pi t) + \cdots$$

$$x(t) = \frac{2}{\pi}Sin(\pi t) - \frac{2}{2\pi}Sin(2\pi t) + \frac{2}{3\pi}Sin(3\pi t) - \frac{2}{4\pi}Sin(4\pi t) + \frac{2}{5\pi}Sin(5\pi t) + \cdots$$

$$x(t) = \frac{2}{\pi}\left(Sin(\pi t) - \frac{1}{2}Sin(2\pi t) + \frac{1}{3}Sin(3\pi t) - \frac{1}{4}Sin(4\pi t) + \frac{1}{5}Sin(5\pi t) + \cdots\right)$$

(14.3) Determine the coefficients of exponential Fourier Series (C_0 & C_n) for a periodic signal x(t) = t, over the range 0 < t < 1 with a fundamental time period of T = 1.

Given

x(t) = t, 0 < t < 1 & T=1

$$\Rightarrow \; t_0 = 0,\; t_0 + T = 1, T = 1, w_0 = \frac{2\pi}{T} = 2\pi$$

Coefficient of Exponential Fourier Series

$$C_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)dt$$

$$= \frac{1}{1}\int_{0}^{1} tdt$$

$$= \frac{t^2}{2}\Big|_0^1$$

$$= \frac{1-0}{2}$$

$$= \frac{1}{2}$$

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jnw_0t}dt$$

$$= \frac{1}{1}\int_{0}^{1} te^{-j2n\pi t}dt$$

$$= \frac{te^{-j2n\pi t}}{-j2n\pi} - \int_0^1 (1)\frac{e^{-j2n\pi t}}{(-j2n\pi)}dt$$

$$= \frac{te^{-j2n\pi t}}{-j2n\pi} - \frac{e^{-j2n\pi t}}{(j2n\pi)^2}\Big|_0^1$$

$$= \frac{e^{-j2n\pi} - 0}{-j2n\pi} - \frac{e^{-j2n\pi} - 1}{(j2n\pi)^2}$$

$$= \frac{1}{-j2n\pi}$$

$$= \frac{j}{2n\pi}$$

(14.4) Determine the coefficient of Exponential Fourier Series C_n for a continuous time periodic signal $x(t) = 1 - 2|t|, for|t| < 1$.

$$Given\ t_0 = -1,\ t_0 + T = 1, T = 2, w_0 = \frac{2\pi}{T} = \pi$$

Coefficients of Exponential Fourier Series

$$C_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)dt$$

$$= \frac{1}{2}\int_{-1}^{1}(1 - 2|t|)dt\,;\ |t| = \begin{cases} -t, & t < 0 \\ t, & t \geq 0 \end{cases}$$

$$= \frac{1}{2}\left(\int_{-1}^{0}(1 + 2t)dt + \int_{0}^{1}(1 - 2t)dt\right)$$

$$= \frac{1}{2}\left((t + t^2)\Big|_{-1}^{0} + (t - t^2)\Big|_{0}^{1}\right)$$

$$= \frac{1}{2}(0 + 1 - 1 + 1 - 1 - 0) = 0$$

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jn\pi t}dt$$

$$= \frac{1}{2}\int_{-1}^{1}(1 - 2|t|)e^{-jn\pi t}dt$$

$$= \frac{1}{2}\left(\int_{-1}^{0}(1 + 2t)e^{-jn\pi t}dt + \int_{0}^{1}(1 - 2t)e^{-jn\pi t}dt\right)$$

$$= \frac{1}{2}\left(\frac{(1 + 2t)e^{-jn\pi t}}{-jn\pi} - \frac{2e^{-jn\pi t}}{(jn\pi)^2}\right)\Bigg|_{-1}^{0} + \frac{1}{2}\left(\frac{(1 - 2t)e^{-jn\pi t}}{-jn\pi} + \frac{2e^{-jn\pi t}}{(jn\pi)^2}\right)\Bigg|_{0}^{1}$$

$$= \frac{1}{2}\left(\frac{1 + e^{jn\pi}}{-jn\pi} - \frac{2(1 - e^{jn\pi})}{(jn\pi)^2}\right) + \frac{1}{2}\left(\frac{-e^{-jn\pi} - 1}{-jn\pi} + \frac{2(e^{-jn\pi} - 1)}{(jn\pi)^2}\right)$$

$$= \frac{1}{2}\left(\frac{1 + e^{jn\pi} - e^{-jn\pi} - 1}{-jn\pi}\right) + \frac{1}{2}\left(\frac{2e^{-jn\pi} - 2 - 2 + 2e^{jn\pi}}{(jn\pi)^2}\right)$$

$$= \frac{1}{2}\left(\frac{2jSin(n\pi)}{-jn\pi}\right) + \frac{1}{2}\left(\frac{4Cos(n\pi) - 4}{(jn\pi)^2}\right)$$

$$= 2\left(\frac{Cos(n\pi) - 1}{(jn\pi)^2}\right)$$

$$= 2\left(\frac{1 - Cos(n\pi)}{n^2\pi^2}\right)$$

$$= \begin{cases} \frac{4}{n^2\pi^2}, & n: odd \\ 0, & n: even \end{cases}$$

(14.5) Determine the Fourier Series coefficients a_n, b_n and C_n of $x(t) = 2Sin(w_0t)$.

We know the Trigonometric Fourier Series expansion of a periodic signal x(t) is

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n Cos(nw_0t) + b_n Sin(nw_o t)\right)$$

$$\Rightarrow 2Sin(w_0t) = a_0 + a_1Cos(w_0t) + b_1Sin(w_ot) + \cdots$$

$$\Rightarrow a_0 = a_n = 0$$

$$\Rightarrow b_n = \begin{cases} 2, & n = 1 \\ 0, & oth \end{cases}$$

Coefficients of Exponential Fourier Series

$$C_0 = a_0 = 0$$

$$C_n = \frac{a_n - jb_n}{2} \Rightarrow C_1 = \frac{a_1 - jb_1}{2} = -j$$

$$C_{-n} = \frac{a_n + jb_n}{2} \Rightarrow C_{-1} = \frac{a_1 + jb_1}{2} = j$$

$$\Rightarrow C_n = \begin{cases} j, & n = -1 \\ -j, & n = 1 \\ 0, & oth \end{cases}$$

(14.6) Determine the Fourier Series coefficients a_n, b_n and C_n of

$x(t)= 1+Sin(w_0t)+2Cos(w_0t)+ Cos(2w_0t+\pi/4)$.

We know the Trigonometric Fourier Series expansion of a periodic signal x(t) is

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n Cos(nw_0t) + b_n Sin(nw_o t)\right)$$

$$\Rightarrow 1 + Sin(w_0t) + 2Cos(w_0t) + Cos\left(2w_ot + \frac{\pi}{4}\right) = a_0 + \sum_{n=1}^{\infty}\left(a_n Cos(nw_0t) + b_n Sin(nw_o t)\right)$$

$$\Rightarrow 1 + Sin(w_0t) + 2Cos(w_0t) + \frac{1}{\sqrt{2}}Cos(2w_ot) - \frac{1}{\sqrt{2}}Sin(2w_ot) =$$

$$a_0 + a_1Cos(w_0t) + b_1Sin(w_ot) + a_2Cos(2w_0t) + b_2Sin(2w_ot)$$

$$\Rightarrow a_0 = 1, a_1 = 2, a_2 = \tfrac{1}{\sqrt{2}}, b_1 = 1, b_2 = -\tfrac{1}{\sqrt{2}}$$

$$\Rightarrow a_n = \begin{cases} 1, & n = 0 \\ 2, & n = 1 \\ \frac{1}{\sqrt{2}}, & n = 2 \\ 0, & oth \end{cases}$$

$$\Rightarrow b_n = \begin{cases} 1, & n = 1 \\ -\frac{1}{\sqrt{2}}, & n = 2 \\ 0, & oth \end{cases}$$

Coefficients of Exponential Fourier Series

$$C_0 = a_0 = 1$$

$$C_n = \frac{a_n - jb_n}{2} \Rightarrow C_1 = \frac{a_1 - jb_1}{2} = \frac{2-j}{2} \;\&\; C_{-1} = \frac{a_1 + jb_1}{2} = \frac{2+j}{2}$$

$$C_2 = \frac{a_2 - jb_2}{2} = \frac{1/\sqrt{2} - j(-1/\sqrt{2})}{2} = \frac{1+j}{2\sqrt{2}} \;\&\; C_{-2} = \frac{a_2 + jb_2}{2} = \frac{1/\sqrt{2} + j(-1/\sqrt{2})}{2} = \frac{1-j}{2\sqrt{2}}$$

$$C_n = \begin{cases} 1, & n = 0 \\ \frac{2+j}{2}, & n = -1 \\ \frac{2-j}{2}, & n = 1 \\ \frac{1-j}{2\sqrt{2}}, & n = -2 \\ \frac{1+j}{2\sqrt{2}}, & n = 2 \\ 0, & oth \end{cases}$$

(14.7) Determine the Fourier Series coefficients a_n, b_n and C_n of

$$\boldsymbol{x(t) = 2 + Cos\left(\frac{2\pi}{3}t\right) + 4Sin\left(\frac{5\pi}{3}t\right), w_0 = \frac{\pi}{3}}$$

We know the Trigonometric Fourier Series expansion of a periodic signal x(t) is

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n Cos(nw_0 t) + b_n Sin(nw_o t)\right)$$

$$\Rightarrow 2 + Cos(2w_0 t) + 4Sin(5w_0 t) = a_0 + a_2 Cos(2w_0 t) + b_5 Sin(5w_o t)$$

$$\Rightarrow a_0 = 2, a_2 = 1, b_5 = 4$$

$$\Rightarrow a_n = \begin{cases} 2, & n = 0 \\ 1, & n = 2 \\ 0, & oth \end{cases}$$

$$\Rightarrow b_n = \begin{cases} 4, & n = 5 \\ 0, & oth \end{cases}$$

Coefficients of Exponential Fourier Series

$C_0 = a_0 = 2$

$$C_n = \frac{a_n - jb_n}{2}$$

$$\Rightarrow C_2 = \frac{a_2 - jb_2}{2} = \frac{1}{2},$$

$$\Rightarrow C_{-2} = \frac{a_2 + jb_2}{2} = \frac{1}{2},$$

$$\Rightarrow C_5 = \frac{a_5 - jb_5}{2} = -\frac{4}{2}j = -2j$$

$$\Rightarrow C_{-5} = 2j$$

$$C_n = \begin{cases} 2, & n = 0 \\ \frac{1}{2}, & n = \pm 2 \\ 2j, & n = -5 \\ -2j, & n = 5 \\ 0, & oth \end{cases}$$

(14.8) Evaluate the signal x(t) such that

(i) x(t) is real and odd

(ii) x(t) is periodic with a fundamental period of T=2

(iii) Fourier series coefficient Cn=0 for |n|>1

(iv) $\frac{1}{2}\int_0^2 |x(t)|^2 dt = 1.$

We know the Trigonometric Fourier Series expansion of a periodic signal x(t)

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n Cos(nw_0 t) + b_n Sin(nw_o t)\right)$$

Given that the signal x(t) is odd

$$\Rightarrow x(t) = \sum_{n=1}^{\infty} b_n Sin(nw_o t)$$

$$\Rightarrow x(t) = b_1 Sin(w_o t) + b_2 Sin(2w_o t) + b_3 Sin(3w_o t) + \cdots$$

Given that $C_n = 0$ for $|n| > 1$

$$\Rightarrow x(t) = b_1 Sin(w_o t)$$

Given that the signal x(t) is periodic with the time period T = 2 ⇒ $w_0 = 2\pi/T = \pi$

$\Rightarrow$ x(t) = b_1Sin(πt)

$$Given\ that\ \frac{1}{2}\int_0^2 |x(t)|^2 dt = 1$$

$$\Rightarrow \int_0^2 {b_1}^2 Sin^2(\pi t)dt = 2$$

$$\Rightarrow {b_1}^2 \int_0^2 \frac{1 - Cos(2\pi t)}{2} dt = 2$$

$$\Rightarrow {b_1}^2 \left(t - \frac{Sin(2\pi t)}{2\pi}\right)\Big|_0^2 = 4$$

$$\Rightarrow {b_1}^2(2 - 0) = 4$$

$$\Rightarrow {b_1}^2 = 2$$

$$\Rightarrow b_1 = \pm\sqrt{2}$$

Substitute b_1 in x(t)

$$\Rightarrow x(t) = \pm\sqrt{2}Sin(\pi t)$$

(14.9) If the Trigonometric Fourier Series coefficients of x(t)=1+2Sint+3Cos2t are a_n and b_n, then find $a_0 + a_1 + b_2$.

We know the Trigonometric Fourier Series expansion of a periodic signal x(t) is

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n Cos(nw_0 t) + b_n Sin(nw_o t)\right)$$

$$\Rightarrow 1 + 2Sin(t) + 3Cos(2t) = a_0 + b_1 Sin(w_o t) + a_2 Cos(2w_o t), w_o = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$\Rightarrow 1 + 2Sin(t) + 3Cos(2t) = a_0 + b_1 Sin(t) + a_2 Cos(2t)$$

$$\Rightarrow a_0 = 1, b_1 = 2\ \&\ a_2 = 3$$

$$\Rightarrow a_0 + a_1 + b_2 = 1 + 0 + 0 = 1$$

(14.10) Determine the numerical values (i)T (ii)A, such that one of the component of x(t) is Acos(3πt). Given that the complex exponential representation of a signal x(t) over the interval $(0,T)$ is $x(t) = \sum_{n=-\infty}^{\infty} \frac{3}{4+(n\pi)^2} e^{jn\pi t}$

We know the Exponential Fourier Series (EFS) expansion of a periodic signal x(t) is

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnw_0 t}$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} \frac{3}{4+(n\pi)^2} e^{jn\pi t} = \sum_{n=-\infty}^{\infty} C_n e^{jnw_0 t}$$

(i) Compare the phase

$$\Rightarrow w_0 = \pi \Rightarrow T = \frac{2\pi}{w_0} = \frac{2\pi}{\pi} = 2$$

(ii) Compare the magnitude

$$\Rightarrow C_n = \frac{3}{4+(n\pi)^2}$$

Given that one of the component of x(t) is Acos(3πt)

$$\Rightarrow ACos(3\pi t) = ACos(3w_0 t) = a_n Cos(nw_0 t)$$

$$\Rightarrow n = 3, A = a_n = a_3$$

$$\Rightarrow A = a_n = C_n + C_{-n} = C_3 + C_{-3}$$

$$\Rightarrow A = \frac{3}{4+(3\pi)^2} + \frac{3}{4+(-3\pi)^2} = \frac{3}{4+9\pi^2} + \frac{3}{4+9\pi^2} = \frac{6}{4+9\pi^2}$$

(14.11) Determine the coefficient of exponential Fourier Series C_n for a periodic signal x(t) = t over the range 0 < t < 1 with a fundamental time period of T = 1, hence obtain the summation $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$Given\ \ t_0 = 0,\ t_0 + T = 1, T = 1, w_0 = \frac{2\pi}{T} = 2\pi$$

Determine the coefficient of Exponential Fourier Series

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jnw_0t}dt$$

$$= \frac{1}{1}\int_0^1 te^{-j2n\pi t}dt$$

$$= \frac{te^{-j2n\pi t}}{-j2n\pi} - \frac{e^{-j2n\pi t}}{(j2n\pi)^2}\Big|_0^1$$

$$= \frac{e^{-j2n\pi} - 0}{-j2n\pi} - \frac{e^{-j2n\pi} - 1}{(j2n\pi)^2}$$

$$= \frac{1}{-j2n\pi} = \frac{j}{2n\pi}$$

$$C_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)dt = \frac{1}{1}\int_0^1 tdt = \frac{t^2}{2}\Big|_0^1 = \frac{1-0}{2} = \frac{1}{2}$$

Apply Parseval's theorem to evaluate the summation

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2\,dt = \sum_{n=-\infty}^{\infty} |C_n|^2 = {C_0}^2 + 2\sum_{n=1}^{\infty} |C_n|^2$$

$$\Rightarrow \frac{1}{1}\int_0^1 |t|^2\,dt = {C_0}^2 + 2\sum_{n=1}^{\infty}\left|\frac{j}{2n\pi}\right|^2$$

$$\Rightarrow \frac{t^3}{3}\Big|_0^1 = \left(\frac{1}{2}\right)^2 + 2\sum_{n=1}^{\infty}\frac{1}{4n^2\pi^2}$$

$$\Rightarrow \frac{1}{3} = \frac{1}{4} + \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$$

$$\Rightarrow \frac{1}{2\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right) = \frac{2\pi^2}{12} = \frac{\pi^2}{6}$$

(14.12) Determine the Fourier Transform of unit impulse signal x(t) = δ(t).

From the definition of Fourier Transform

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$\Rightarrow FT[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-jwt}dt$$

Apply the property of impulse signal, δ(t)y(t) = δ(t)y(0)

$$\Rightarrow FT[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{0}dt$$

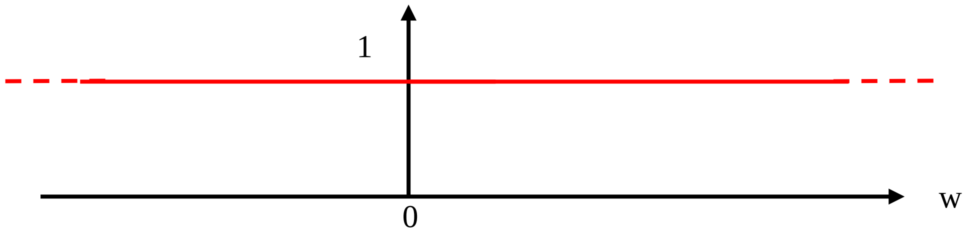
$$= \int_{-\infty}^{\infty} \delta(t)1dt$$

$$= \int_{-\infty}^{\infty} \delta(t)dt$$

Area under unit impulse signal δ(t) is '1'

$$\Rightarrow FT[\delta(t)] = X(w) = 1$$

$$FT[\delta(t)] = X(w) = 1$$



Note: Impulse signal and DC signal (Constant signal) are Fourier transformable pairs.

(14.13) Determine the Fourier Transform of decaying exponential signal $x(t) = e^{-at}u(t), a > 0$. Hence obtain its magnitude and phase spectrum.

From the definition of Fourier Transform

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$\Rightarrow FT[e^{-at}u(t)] = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-jwt}dt;\ u(t) = 1, t > 0$$

$$= \int_{0}^{\infty} e^{-(a+jw)t}dt$$

$$= \frac{e^{-(a+jw)t}}{-(a+jw)}\Bigg|_{0}^{\infty}$$

$$= \frac{e^{-\infty} - e^{0}}{-(a+jw)}$$

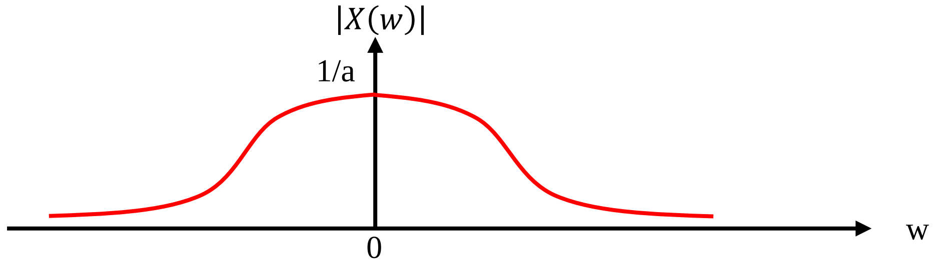
$$= \frac{0-1}{-(a+jw)}$$

$$= \frac{1}{a+jw}$$

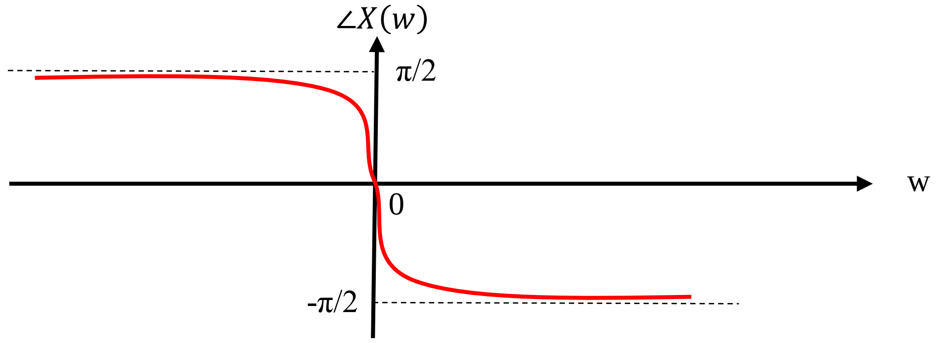
$$\Rightarrow FT[e^{-at}u(t)] = X(w) = \frac{1}{a+jw}$$

Frequency domain is complex, now determine its magnitude and phase

$$Magnitude\ Spectrum:\ |X(w)| = \frac{1}{\sqrt{a^2+w^2}}$$



$$Phase\ Spectrum:\ \angle X(w) = \angle(1+j0) - \angle(a+jw) = 0 - \tan^{-1}\left(\frac{w}{a}\right) = -\tan^{-1}\left(\frac{w}{a}\right)$$



(14.14) Determine the Fourier Transform of raising exponential signal $x(t) = e^{at}u(-t), a > 0$. Hence obtain its magnitude and phase spectrum.

From the definition of Fourier Transform

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$\Rightarrow FT[e^{at}u(-t)] = \int_{-\infty}^{\infty} e^{at}u(-t)e^{-jwt}dt;\ u(-t) = 1, t < 0$$

$$= \int_{-\infty}^{0} e^{(a-jw)t}dt$$

$$= \frac{e^{(a-jw)t}}{a-jw}\Bigg|_{-\infty}^{0}$$

$$= \frac{e^{0} - e^{-\infty}}{a-jw}$$

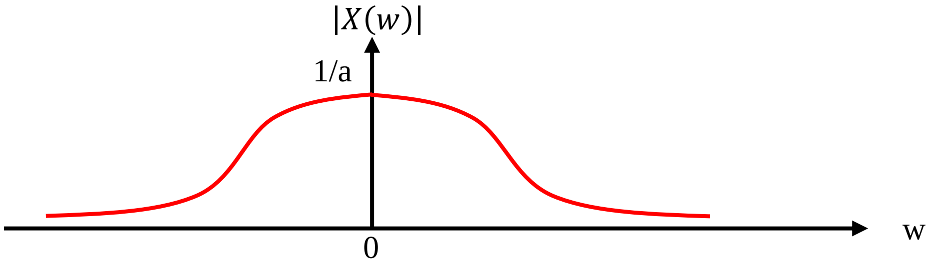
$$= \frac{1-0}{a-jw}$$

$$= \frac{1}{a-jw}$$

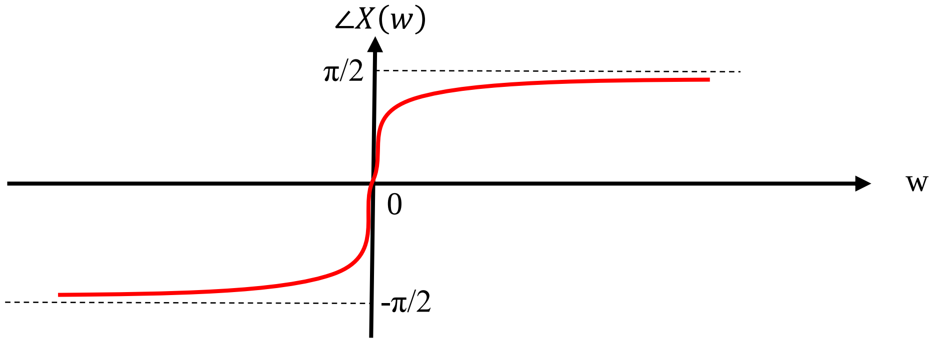
$$\Rightarrow FT[e^{at}u(-t)] = X(w) = \frac{1}{a-jw}$$

Frequency domain is complex, now determine its magnitude and phase

$$Magnitude\ Spectrum:\ |X(w)| = \frac{1}{\sqrt{a^2+w^2}}$$



$$Phase\ Spectrum:\ \angle X(w) = \angle(1+j0) - \angle(a-jw) = 0 - \tan^{-1}\left(\frac{-w}{a}\right) = \tan^{-1}\left(\frac{w}{a}\right)$$



(14.15) Determine the Fourier Transform of double exponential signal $x(t) = e^{-a|t|}, a > 0$.

From the definition of Fourier Transform

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$\Rightarrow FT[e^{-a|t|}] = \int_{-\infty}^{\infty} e^{-a|t|}e^{-jwt}dt$$

$$= \int_{-\infty}^{0} e^{-a(-t)}e^{-jwt}dt + \int_{0}^{\infty} e^{-a(t)}e^{-jwt}dt$$

$$= \int_{-\infty}^{0} e^{at}e^{-jwt}dt + \int_{0}^{\infty} e^{-at}e^{-jwt}dt$$

$$= \int_{-\infty}^{0} e^{(a-jw)t}dt + \int_{0}^{\infty} e^{-(a+jw)t}dt$$

$$= \frac{e^{(a-jw)t}}{a-jw}\Bigg|_{-\infty}^{0} + \frac{e^{-(a+jw)t}}{-(a+jw)}\Bigg|_{0}^{\infty}$$

$$= \frac{e^{0}-e^{-\infty}}{a-jw} + \frac{e^{-\infty}-e^{0}}{-(a+jw)}$$

$$= \frac{1-0}{a-jw} + \frac{0-1}{-(a+jw)}$$

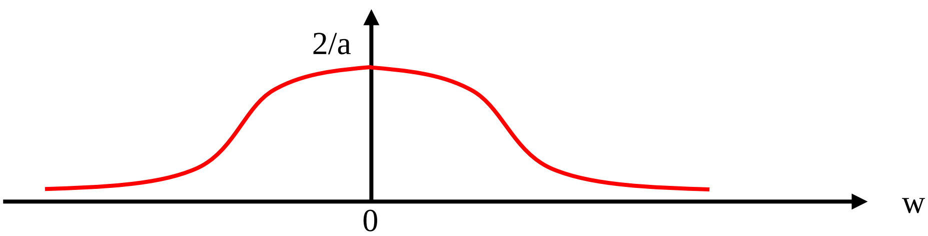
$$= \frac{1}{a-jw} + \frac{1}{a+jw}$$

$$= \frac{a+jw+a-jw}{(a-jw)(a+jw)}$$

$$= \frac{2a}{a^2+w^2}$$

$$\Rightarrow FT[e^{-a|t|}] = X(w) = \frac{2a}{a^2+w^2}$$

$$FT[e^{-a|t|}] = X(w) = \frac{2a}{a^2+w^2}$$



(14.16) Find the Fourier Transform of a signal $x(t) = Sgn(t) = \underset{a \to 0}{Lt} e^{-a|t|} Sgn(t), a > 0.$

From the definition of Fourier Transform

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$$

$$\Rightarrow FT\left[\underset{a \to 0}{Lt} e^{-a|t|} Sgn(t)\right] = \int_{-\infty}^{\infty} \underset{a \to 0}{Lt} e^{-a|t|} Sgn(t) e^{-jwt} dt$$

$$= \underset{a \to 0}{Lt} \int_{-\infty}^{\infty} e^{-a|t|} Sgn(t) e^{-jwt} dt$$

$$= \underset{a \to 0}{Lt} \left(\int_{-\infty}^{0} e^{-a(-t)} (-1) e^{-jwt} dt + \int_{0}^{\infty} e^{-a(t)} (1) e^{-jwt} dt \right)$$

$$= \underset{a \to 0}{Lt} \left(-\int_{-\infty}^{0} e^{at} e^{-jwt} dt + \int_{0}^{\infty} e^{-at} e^{-jwt} dt \right)$$

$$= \underset{a \to 0}{Lt} \left(-\int_{-\infty}^{0} e^{(a-jw)t} dt + \int_{0}^{\infty} e^{-(a+jw)t} dt \right)$$

$$= \underset{a \to 0}{Lt} \left(-\frac{e^{(a-jw)t}}{a - jw} \Bigg|_{-\infty}^{0} + \frac{e^{-(a+jw)t}}{-(a+jw)} \Bigg|_{0}^{\infty} \right)$$

$$= \underset{a \to 0}{Lt} \left(-\frac{e^{0} - e^{-\infty}}{a - jw} + \frac{e^{-\infty} - e^{0}}{-(a+jw)} \right)$$

$$= \underset{a \to 0}{Lt} \left(-\frac{1 - 0}{a - jw} + \frac{0 - 1}{-(a+jw)} \right)$$

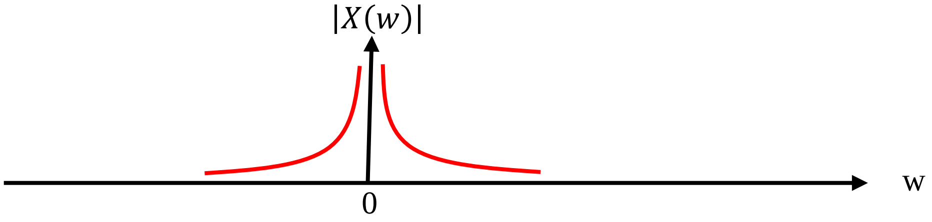
$$= \underset{a \to 0}{Lt} \left(-\frac{1}{a - jw} + \frac{1}{a + jw} \right)$$

$$= \frac{1}{jw} + \frac{1}{jw}$$

$$= \frac{2}{jw}$$

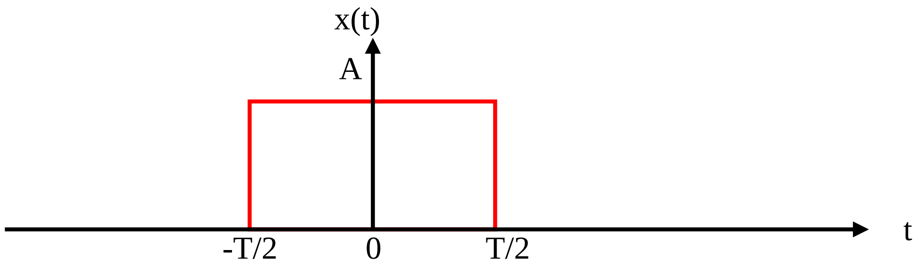
$$\Rightarrow FT\left[\underset{a \to 0}{Lt} e^{-a|t|} Sgn(t)\right] = FT[Sgn(t)] = X(w) = \frac{2}{jw} = \frac{1}{j\pi f}$$

$$Magnitude\ Spectrum: \ |X(w)| = \left|\frac{2}{jw}\right| = \left|\frac{2}{w}\right|$$



(14.17) Determine the Fourier Transform of rectangular signal $x(t) = Arect\left(\frac{t}{T}\right)$.

$$We\ know, x(t) = A.rect\left(\frac{t}{T}\right) = A.G_T(t) = \begin{cases} A, -\frac{T}{2} < t < \frac{T}{2} \\ 0, Otherwise \end{cases}$$



From the definition of Fourier Transform

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$\Rightarrow FT\left[Arect\left(\frac{t}{T}\right)\right] = \int_{-\infty}^{\infty} Arect\left(\frac{t}{T}\right)e^{-jwt}dt$$

$$= \int_{-T/2}^{T/2} Ae^{-jwt}dt$$

$$= A\frac{e^{-jwt}}{-jw}\Bigg|_{-T/2}^{T/2}$$

$$= A\frac{e^{-jwT/2} - e^{jwT/2}}{-jw}$$

$$= A\frac{e^{jwT/2} - e^{-jwT/2}}{jw};\ e^{j\theta} - e^{-j\theta} = 2jSin(\theta)$$

$$= A\frac{2jSin\left(\frac{wT}{2}\right)}{jw}$$

$$= \frac{2A}{w}Sin\left(\frac{wT}{2}\right) = \frac{2A}{w}\left(\frac{wT}{2}\right)\frac{Sin\left(\frac{wT}{2}\right)}{wT/2} = AT\frac{Sin\left(\frac{wT}{2}\right)}{wT/2} = ATSa\left(\frac{wT}{2}\right)$$

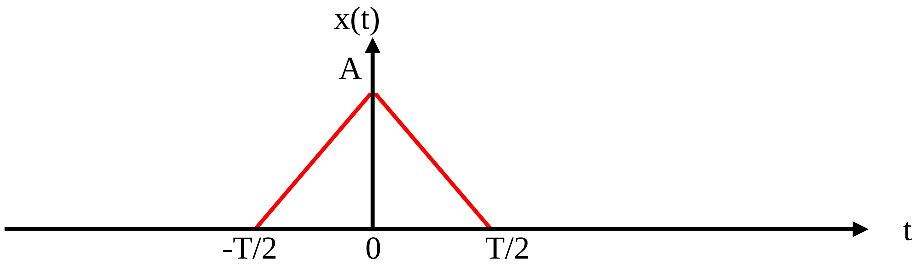
$$= \frac{2A}{2\pi f}Sin\left(\frac{2\pi fT}{2}\right) = \frac{A}{\pi f}Sin(\pi fT) = AT\frac{Sin(\pi fT)}{\pi fT} = ATSinc(fT)$$

$$\Rightarrow FT\left[Arect\left(\frac{t}{T}\right)\right] = ATSinc(fT) = ATSa\left(\frac{wT}{2}\right)$$

Note: Rectangular signal and Sinc or Sampling signals are Fourier transformable pairs.

(14.18) Determine the Fourier Transform of triangular signal $x(t) = A.tri\left(\frac{t}{T}\right)$

$$We\ know, x(t) = A.tri\left(\frac{t}{T}\right) = \begin{cases} A\left(1 - \frac{2}{T}|t|\right), -\frac{T}{2} < t < \frac{T}{2} \\ 0, \qquad\qquad other\ wise \end{cases}$$



From the definition of Fourier Transform

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$\Rightarrow FT\left[A.tri\left(\frac{t}{T}\right)\right] = \int_{-\infty}^{\infty} A.tri\left(\frac{t}{T}\right)e^{-jwt}dt$$

$$= \int_{-T/2}^{T/2} A\left(1 - \frac{2}{T}|t|\right)e^{-jwt}dt$$

$$= A\left(\int_{-T/2}^{0}\left(1 - \frac{2}{T}(-t)\right)e^{-jwt}dt + \int_{0}^{T/2}\left(1 - \frac{2}{T}(t)\right)e^{-jwt}dt\right)$$

$$= A\left(\int_{-T/2}^{0}\left(1 + \frac{2}{T}t\right)e^{-jwt}dt + \int_{0}^{T/2}\left(1 - \frac{2}{T}t\right)e^{-jwt}dt\right)$$

$$= A\left(\left(1 + \frac{2}{T}t\right)\frac{e^{-jwt}}{-jw} - \frac{2}{T}\frac{e^{-jwt}}{(-jw)^2}\Bigg|_{-T/2}^{0} + \left(1 - \frac{2}{T}t\right)\frac{e^{-jwt}}{-jw} + \frac{2}{T}\frac{e^{-jwt}}{(-jw)^2}\Bigg|_{0}^{T/2}\right)$$

$$= A\left(\left(\frac{1}{-jw} - 0\right) - \frac{2}{T}\frac{\left(1 - e^{\frac{jwT}{2}}\right)}{(-jw)^2} + \left(0 - \frac{1}{-jw}\right) + \frac{2}{T}\frac{\left(e^{-\frac{jwT}{2}} - 1\right)}{(-jw)^2}\right)$$

$$= A\left(-\frac{1}{jw} + \frac{2}{Tw^2}\left(1 - e^{\frac{jwT}{2}}\right) + \frac{1}{jw} - \frac{2}{Tw^2}\left(e^{-\frac{jwT}{2}} - 1\right)\right)$$

$$= \frac{2A}{Tw^2}\left(1 - e^{\frac{jwT}{2}} - e^{-\frac{jwT}{2}} + 1\right)$$

$$= \frac{2A}{Tw^2}\left(2 - \left(e^{\frac{jwT}{2}} + e^{-\frac{jwT}{2}}\right)\right)$$

$$= \frac{2A}{Tw^2}\left(2 - 2Cos\frac{wT}{2}\right)$$

$$= \frac{4A}{Tw^2}\left(1 - Cos\frac{wT}{2}\right); 1 - Cos(\theta) = 2Sin^2(\theta/2)$$

$$= \frac{4A}{Tw^2}\left(2Sin^2\frac{wT}{4}\right)$$

$$= \frac{8A}{Tw^2}Sin^2\frac{wT}{4} = \frac{AT}{2}\frac{Sin^2\frac{wT}{4}}{\left(\frac{wT}{4}\right)^2} = \frac{AT}{2}Sa^2\left(\frac{wT}{4}\right)$$

$$= \frac{8A}{T4\pi^2f^2}Sin^2\frac{2\pi fT}{4} = \frac{2A}{T\pi^2f^2}Sin^2\frac{\pi fT}{2} = \frac{AT}{2}\frac{Sin^2\frac{\pi fT}{2}}{\left(\frac{\pi fT}{2}\right)^2} = \frac{AT}{2}Sinc^2\left(\frac{fT}{2}\right)$$

$$\Rightarrow FT\left[A.tri\left(\frac{t}{T}\right)\right] = \frac{AT}{2}Sa^2\left(\frac{wT}{4}\right) = \frac{AT}{2}Sinc^2\left(\frac{fT}{2}\right)$$

Note: Triangular signal and Sinc square or Sampling square signals are Fourier transformable pairs.

(14.19) Determine the Fourier Transform of rectangular signal $x(t) = rect(2t)$

We know that,

$$FT\left[Arect\left(\frac{t}{T}\right)\right] = ATSinc(fT) = ATSa\left(\frac{wT}{2}\right)$$

Put, A = 1 and T = 1/2

$$FT[rect(2t)] = \frac{1}{2}Sinc\left(\frac{f}{2}\right) = \frac{1}{2}Sa\left(\frac{w}{4}\right) = \frac{1}{2}\frac{Sin\left(\pi\frac{f}{2}\right)}{\pi f/2} = \frac{Sin\left(\frac{\pi f}{2}\right)}{\pi f}$$

(14.20) Determine the Fourier Transform of rectangular signal $x(t) = rect(t/2)$

We know that,

$$FT\left[Arect\left(\frac{t}{T}\right)\right] = ATSinc(fT) = ATSa\left(\frac{wT}{2}\right)$$

Put, A = 1 and T = 2

$$FT\left[rect\left(\frac{t}{2}\right)\right] = 2Sinc(2f) = 2Sa(w)$$

(14.21) Determine the Fourier Transform of a Gaussian pulse (signal) $x(t) = e^{-kt^2}, k > 0$

From the definition of Fourier Transform

$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$\Rightarrow FT\left[e^{-kt^2}\right] = \int_{-\infty}^{\infty} e^{-kt^2} e^{-jwt}dt$$

$$= \int_{-\infty}^{\infty} e^{-(kt^2+jwt)}dt; kt^2 + jwt = a^2 + 2ab = a^2 + 2ab + b^2 - b^2, a = \sqrt{k}t, 2ab = jwt$$

$$= \int_{-\infty}^{\infty} e^{-\left((\sqrt{k}t)^2+2(\sqrt{k}t)\left(\frac{jw}{2\sqrt{k}}\right)+\left(\frac{jw}{2\sqrt{k}}\right)^2-\left(\frac{jw}{2\sqrt{k}}\right)^2\right)}dt, b = \frac{jwt}{2a} = \frac{jwt}{2\sqrt{k}t} = \frac{jw}{2\sqrt{k}}$$

$$= \int_{-\infty}^{\infty} e^{-\left(\sqrt{k}t+\frac{jw}{2\sqrt{k}}\right)^2} e^{\left(\frac{jw}{2\sqrt{k}}\right)^2} dt$$

$$= e^{\left(\frac{jw}{2\sqrt{k}}\right)^2} \int_{-\infty}^{\infty} e^{-\left(\sqrt{k}t+\frac{jw}{2\sqrt{k}}\right)^2} dt, \sqrt{k}t + \frac{jw}{2\sqrt{k}} = \tau \Rightarrow \sqrt{k}dt = d\tau$$

$$= e^{-\frac{w^2}{4k}} \int_{-\infty}^{\infty} e^{-\tau^2} \left(\frac{d\tau}{\sqrt{k}}\right)$$

$$= \frac{1}{\sqrt{k}} e^{-\frac{w^2}{4k}} \int_{-\infty}^{\infty} e^{-\tau^2} d\tau$$

$$= \frac{1}{\sqrt{k}} e^{-\frac{w^2}{4k}} (\sqrt{\pi})$$

$$= \sqrt{\frac{\pi}{k}} e^{-\frac{w^2}{4k}}$$

$$\Rightarrow FT\left[e^{-kt^2}\right] = \sqrt{\frac{\pi}{k}} e^{-\frac{w^2}{4k}} = \sqrt{\frac{\pi}{k}} e^{-\frac{\pi^2 f^2}{k}}$$

Note:

- $If\ k = \pi, then\ FT\left[e^{-\pi t^2}\right] = e^{-\pi f^2}$
- Fourier Transform of a Gaussian pulse is Gaussian pulse only.

(14.22) Find G(0), such that FT[g(t)]=G(f), where g(t) is Gaussian pulse $g(t) = e^{-kt^2}, k > 0$.

$$We\ know\ that, FT[g(t)] = G(f) = FT\left[e^{-kt^2}\right] = \sqrt{\frac{\pi}{k}} e^{-\frac{\pi^2 f^2}{k}}$$

$$\Rightarrow G(0) = \sqrt{\frac{\pi}{k}} e^{-0} = \sqrt{\frac{\pi}{k}}$$

(14.23) Find the constant k, such that $\frac{d}{df}G(f) = -afG(f)$, where G(f) is the Fourier Transform of the Gaussian pulse $g(t) = e^{-kt^2}, k > 0$.

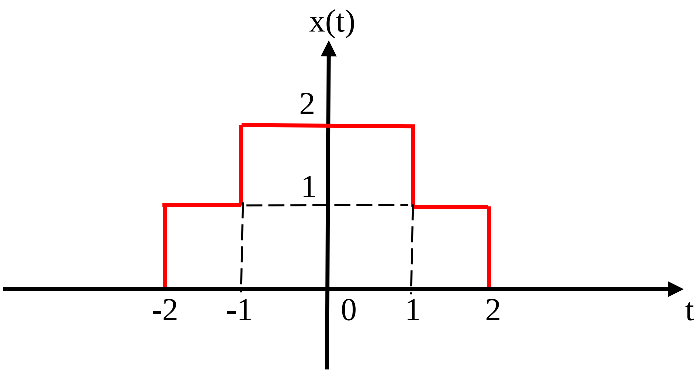
$$FT[g(t)] = G(f) = FT\left[e^{-kt^2}\right] = \sqrt{\frac{\pi}{k}}e^{-\frac{\pi^2 f^2}{k}}$$

$$\Rightarrow \frac{d}{df}G(f) = \sqrt{\frac{\pi}{k}}e^{-\frac{\pi^2 f^2}{k}}\left(-\frac{\pi^2 2f}{k}\right)$$

$$\Rightarrow -afG(f) = G(f)\left(-\frac{\pi^2 2f}{k}\right)$$

$$\Rightarrow a = \frac{2\pi^2}{k} \Rightarrow k = \frac{2\pi^2}{a}$$

(14.24) Find the Fourier Transform of x(t)



$$FT[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt$$

$$= \int_{-2}^{-1} 1e^{-jwt}dt + \int_{-1}^{1} 2e^{-jwt}dt + \int_{1}^{2} 1e^{-jwt}dt$$

$$= \frac{e^{-jwt}}{-jw}\Big|_{-2}^{-1} + 2\frac{e^{-jwt}}{-jw}\Big|_{-1}^{1} + \frac{e^{-jwt}}{-jw}\Big|_{1}^{2}$$

$$= \frac{e^{jw} - e^{j2w} + 2(e^{-jw} - e^{jw}) + e^{-j2w} - e^{-jw}}{-jw}$$

$$= \frac{-e^{jw} + e^{j2w} - 2e^{-jw} + 2e^{jw} - e^{-j2w} + e^{-jw}}{jw}$$

$$= \frac{e^{jw} - e^{-jw} + e^{j2w} - e^{-j2w}}{jw}$$

$$= \frac{2jSin(w) + 2jSin(2w)}{jw} = \frac{2\big(Sin(w) + Sin(2w)\big)}{w}$$

$$= 2\frac{Sin(w)}{w} + 4\frac{Sin(2w)}{2w} = 2Sa(w) + 4Sa(2w)$$

(14.25) Determine the Fourier Transform of a signal $x(t) = 2\delta(t) - 3e^{-2t}u(t)$.

$$FT[x(t)] = FT[2\delta(t) - 3e^{-2t}u(t)]$$

$$= 2FT[\delta(t)] - 3FT[e^{-2t}u(t)]; FT[\delta(t)] = 1 \ \& \ FT[e^{-at}u(t)] = \frac{1}{a + jw}$$

$$= 2(1) - 3\left(\frac{1}{2 + jw}\right)$$

$$= 2 - \frac{3}{2 + jw}$$

$$= \frac{4 + j2w - 3}{2 + jw}$$

$$= \frac{1 + j2w}{2 + jw}$$

$$\Rightarrow FT[x(t)] = FT[2\delta(t) - 3e^{-2t}u(t)] = \frac{1 + j2w}{2 + jw}$$

$$Magnitude\ Spectrum: \ |X(w)| = \sqrt{\frac{1^2 + (2w)^2}{2^2 + w^2}} = \sqrt{\frac{1 + 4w^2}{4 + w^2}}$$

$Phase\ Spectrum:$

$$\angle X(w) = \tan^{-1}(2w) - \tan^{-1}\left(\frac{w}{2}\right) = \tan^{-1}\left(\frac{2w - \frac{w}{2}}{1 + 2w(\frac{w}{2})}\right) = \tan^{-1}\left(\frac{3w}{2(1 + w^2)}\right)$$

(14.26) Determine the Fourier Transform of a signal $x(t) = 3e^{-2t}u(t) - 2e^{3t}u(-t)$

$$FT[x(t)] = FT[3e^{-2t}u(t) - 2e^{3t}u(-t)]$$

$$= 3FT[e^{-2t}u(t)] - 2FT[e^{3t}u(-t)], FT[e^{-at}u(t)] = \frac{1}{a + jw} \ \& \ FT[e^{at}u(-t)] = \frac{1}{a - jw}$$

$$= 3\left(\frac{1}{2 + jw}\right) - 2\left(\frac{1}{3 - jw}\right)$$

$$= \frac{3}{2 + jw} - \frac{2}{3 - jw}$$

$$= \frac{9 - j3w - 4 - j2w}{(2 + jw)(3 - jw)}$$

$$= \frac{5 - j5w}{(2 + jw)(3 - jw)}$$

$$= \frac{5(1 - jw)}{(2 + jw)(3 - jw)}$$

$$\Rightarrow FT[x(t)] = FT[3e^{-2t}u(t) - 2e^{3t}u(-t)] = \frac{5(1 - jw)}{(2 + jw)(3 - jw)}$$

$$Magnitude\ Spectrum: \ |X(w)| = 5\sqrt{\frac{1+w^2}{(4+w^2)(9+w^2)}}$$

$Phase\ Spectrum:$

$$\angle X(w) = \tan^{-1}(-w) - \tan^{-1}\left(\frac{w}{2}\right) - \tan^{-1}\left(\frac{-w}{3}\right) = -\tan^{-1}(w) - \tan^{-1}\left(\frac{w}{2}\right) + \tan^{-1}\left(\frac{w}{3}\right)$$

(14.27) Determine the Fourier Transform of a signal $y(t) = 3e^{-2t}u(t-9)$.

$$We\ know\ that, FT[e^{-at}u(t)] = \frac{1}{a+jw}$$

$$\Rightarrow FT[e^{-2t}u(t)] = \frac{1}{2+jw}$$

$$Apply\ time\ shifting\ property, FT[x(t-t_0)] = e^{-jwt_0}X(w)$$

$$\Rightarrow FT\left[e^{-2(t-9)}u(t-9)\right] = e^{-jw(9)}\frac{1}{2+jw}$$

$$\Rightarrow FT[e^{-2t}e^{18}u(t-9)] = \frac{e^{-j9w}}{2+jw}$$

$$\Rightarrow FT[e^{-2t}u(t-9)] = \frac{e^{-j9w}e^{-18}}{2+jw}$$

$$\Rightarrow FT[3e^{-2t}u(t-9)] = FT[y(t)] = \frac{3e^{-9(2+jw)}}{2+jw}$$

(14.28) Determine the Fourier Transform of a signal $y(t) = e^{j4t}x(t), given\ x(t) = e^{-3|t|}$.

$$FT[y(t)] = FT\left[e^{j4t}x(t)\right]; \ FT[e^{jw_0t}x(t)] = X(w-w_0)$$

$$= X(w-4); X(w) = FT[x(t)] = FT\left[e^{-3|t|}\right] = \frac{2\times 3}{3^2+w^2} = \frac{6}{9+w^2}$$

$$= \frac{6}{9+(w-4)^2}$$

$$= \frac{6}{9+w^2+16-8w}$$

$$= \frac{6}{w^2-8w+25}$$

(14.29) Determine the Fourier Transform of a signal $y(t) = e^{j4t}x(t), given\ x(t) = 3e^{-2t}u(t)$.

$$FT[y(t)] = FT\left[e^{j4t}x(t)\right];\ \ FT[e^{jw_0t}x(t)] = X(w - w_0)$$

$$= X(w-4); X(w) = FT[x(t)] = FT[3e^{-2t}u(t)] = \frac{3}{2+jw}$$

$$= \frac{3}{2+j(w-4)}$$

(14.30) Determine the Fourier Transform of a signal $y(t) = e^{-2t}Cos(4t)u(t)$.

$$FT[y(t)] = FT[e^{-2t}Cos(4t)u(t)]$$

$$= FT\left[e^{-2t}\frac{e^{j4t}+e^{-j4t}}{2}u(t)\right]$$

$$= \frac{1}{2}FT\left[e^{j4t}e^{-2t}u(t) + e^{-j4t}e^{-2t}u(t)\right]; let\ e^{-2t}u(t) = x(t)$$

$$= \frac{1}{2}FT\left[e^{j4t}x(t) + e^{-j4t}x(t)\right]; FT[e^{jw_0t}x(t)] = X(w-w_0)$$

$$= \frac{1}{2}FT[X(w-4) + X(w+4)]; X(w) = \frac{1}{2+jw}$$

$$= \frac{1}{2}\left(\frac{1}{2+j(w-4)} + \frac{1}{2+j(w+4)}\right)$$

$$= \frac{2+j(w+4)+2+j(w-4)}{2\big(2+j(w-4)\big)\big(2+j(w+4)\big)}$$

$$= \frac{4+j2w}{2\big(4-(w-4)(w+4)+j2(w+4+w-4)\big)}$$

$$= \frac{2(2+jw)}{2(4-(w^2-16)+j2(2w))}$$

$$= \frac{2+jw}{20-w^2+j4w}$$

(14.31) Find the Fourier Transform of a signal $y(t) = x(t) + x(-t), given\ x(t) = e^{-3t}u(t)$.

$$FT[y(t)] = FT[x(t) + x(-t)]$$

$$= FT[x(t)] + FT[x(-t)]$$

$$= X(w) + X(-w); X(w) = FT[x(t)] = FT[e^{-3t}u(t)] = \frac{1}{3+jw}$$

$$= \frac{1}{3+jw} + \frac{1}{3-jw}$$

$$= \frac{3-jw+3+jw}{(3+jw)(3-jw)} = \frac{6}{9+w^2}$$

(14.32) If x(t) is real signal and FT[x(t)]=X(w), then show that

$$(a) X^*(w) = X(-w)\ (b) X_e(w) = ReP\{X(w)\}\ (c) X_o(w) = jImP\{X(w)\}$$

(a) *From the conjugate property of Fourier Transform*

$FT[x^*(t)] = X^*(-w);$ *if* $x(t)$ *is real then* $x^*(t) = x(t)$

$$\Rightarrow FT[x(t)] = X^*(-w)$$

$$\Rightarrow X(w) = X^*(-w)$$

$$\Rightarrow X^*(w) = X(-w)$$

(b) *Even part of* $x(t)$ *can be computed from*

$$x_e(t) = \frac{x(t) + x(-t)}{2}$$

$$\Rightarrow FT[x_e(t)] = FT\left[\frac{x(t) + x(-t)}{2}\right]$$

$$\Rightarrow X_e(w) = \frac{FT[x(t)] + FT[x(-t)]}{2}$$

$$= \frac{X(w) + X(-w)}{2}$$

$$= \frac{X(w) + X^*(w)}{2}$$

$$= ReP\{X(w)\}$$

(c) *Odd part of* $x(t)$ *can be computed from*

$$x_o(t) = \frac{x(t) - x(-t)}{2}$$

$$\Rightarrow FT[x_o(t)] = FT\left[\frac{x(t) - x(-t)}{2}\right]$$

$$\Rightarrow X_o(w) = \frac{FT[x(t)] - FT[x(-t)]}{2}$$

$$= \frac{X(w) - X(-w)}{2}$$

$$= \frac{X(w) - X^*(w)}{2}$$

$$= jImP\{X(w)\}$$

(14.33) Determine the Fourier Transform of y(t) = x(2t-6), given that FT[x(t)]=X(w).

$$FT[y(t)] = FT[x(2t-6)]$$
$$= FT\big[x\big(2(t-3)\big)\big];\ \textit{time shifting property}\ \mathrm{FT}[\,x(t-t_0)\,] = e^{-jwt_0}\,X(w)$$
$$= e^{-j3w}\,FT[x(2t)];\ \textit{time scaling property}\ FT[x(at)] = \frac{1}{|a|}X\left(\frac{w}{a}\right)$$
$$= e^{-j3w}\left(\frac{1}{2}\ X\left(\frac{w}{2}\right)\right)$$
$$= \frac{1}{2}e^{-j3w}\,X\left(\frac{w}{2}\right)$$

(14.34) Determine the Fourier Transform of y(t) = x(2t+6) + x(2t-6)

$$FT[y(t)] = FT[x(2t+6)+x(2t-6)]$$
$$= FT\big[x\big(2(t+3)\big)+x\big(2(t-3)\big)\big]$$
$$= e^{j3w}\,FT[x(2t)] + e^{-j3w}\,FT[x(2t)]$$
$$= e^{j3w}\left(\frac{1}{2}\ X\left(\frac{w}{2}\right)\right) + e^{-j3w}\left(\frac{1}{2}\ X\left(\frac{w}{2}\right)\right)$$
$$= \frac{1}{2}e^{j3w}\,X\left(\frac{w}{2}\right) + \frac{1}{2}e^{-j3w}\,X\left(\frac{w}{2}\right)$$
$$= \frac{e^{j3w}+e^{-j3w}}{2}\,X\left(\frac{w}{2}\right)$$
$$= Cos(3w)\,X\left(\frac{w}{2}\right)$$

(14.35) Determine the Fourier Transform of y(t) = x(2t+6) - x(2t-6)

$$FT[y(t)] = FT[x(2t+6)-x(2t-6)]$$
$$= FT\big[x\big(2(t+3)\big)-x\big(2(t-3)\big)\big]$$
$$= e^{j3w}\,FT[x(2t)] - e^{-j3w}\,FT[x(2t)]$$
$$= e^{j3w}\left(\frac{1}{2}\ X\left(\frac{w}{2}\right)\right) - e^{-j3w}\left(\frac{1}{2}\ X\left(\frac{w}{2}\right)\right)$$
$$= \frac{1}{2}e^{j3w}\,X\left(\frac{w}{2}\right) - \frac{1}{2}e^{-j3w}\,X\left(\frac{w}{2}\right)$$
$$= j\,\frac{e^{j3w}-e^{-j3w}}{2j}\,X\left(\frac{w}{2}\right)$$
$$= jSin(3w)\,X\left(\frac{w}{2}\right)$$

(214.36) Determine the Fourier Transform of y(t) = x(2t+3) + x(2t-3) + x(-2t+3) + x(-2t-3), given FT[x(t)]=X(w).

$$FT[y(t)] = FT[\mathrm{x}(2\mathrm{t}+3) + \mathrm{x}(2\mathrm{t}-3) + \mathrm{x}(-2\mathrm{t}+3) + \mathrm{x}(-2\mathrm{t}-3)]$$

$$= FT\left[\mathrm{x}\left(2\left(\mathrm{t}+\frac{3}{2}\right)\right) + \mathrm{x}\left(2\left(\mathrm{t}-\frac{3}{2}\right)\right) + \mathrm{x}\left(-2\left(\mathrm{t}-\frac{3}{2}\right)\right) + \mathrm{x}\left(-2\left(\mathrm{t}+\frac{3}{2}\right)\right)\right]$$

$$= e^{j\frac{3}{2}w}\, FT[x(2t)] + e^{-j\frac{3}{2}w}\, FT[x(2t)] + e^{-j\frac{3}{2}w}\, FT[x(-2t)] + e^{j\frac{3}{2}w}\, FT[x(-2t)]$$

$$= e^{j\frac{3}{2}w}\left(\frac{1}{2}X\left(\frac{w}{2}\right)\right) + e^{-j\frac{3}{2}w}\left(\frac{1}{2}X\left(\frac{w}{2}\right)\right) + e^{-j\frac{3}{2}w}\left(\frac{1}{2}X\left(\frac{w}{-2}\right)\right) + e^{j\frac{3}{2}w}\left(\frac{1}{2}X\left(\frac{w}{-2}\right)\right)$$

$$= \frac{e^{j\frac{3}{2}w}+e^{-j\frac{3}{2}w}}{2}X\left(\frac{w}{2}\right) + \frac{e^{j\frac{3}{2}w}+e^{-j\frac{3}{2}w}}{2}X\left(-\frac{w}{2}\right)$$

$$= Cos\left(\frac{3}{2}w\right)X\left(\frac{w}{2}\right) + Cos\left(\frac{3}{2}w\right)X\left(-\frac{w}{2}\right)$$

$$= Cos\left(\frac{3}{2}w\right)\left[X\left(\frac{w}{2}\right) + X\left(-\frac{w}{2}\right)\right]$$

(14.37) Determine the Fourier Transform of y(t) = x(2t+6)+x(2t-6), $\boldsymbol{given\ x(t) = e^{-3|t|}}$**.**

$$FT[y(t)] = FT[x(2t+6) + x(2t-6)]$$

$$= FT\big[x\big(2(t+3)\big) + x\big(2(t-3)\big)\big]$$

$$= e^{j3w}\, FT[x(2t)] + e^{-j3w}\, FT[x(2t)]$$

$$= e^{j3w}\left(\frac{1}{2}\ X\left(\frac{w}{2}\right)\right) + e^{-j3w}\left(\frac{1}{2}\ X\left(\frac{w}{2}\right)\right)$$

$$= \frac{1}{2}e^{j3w}\,X\left(\frac{w}{2}\right) + \frac{1}{2}e^{-j3w}\,X\left(\frac{w}{2}\right)$$

$$= \frac{e^{j3w}+e^{-j3w}}{2}\ X\left(\frac{w}{2}\right);\ X(w) = FT[x(t)] = FT\left[e^{-3|t|}\right] = \frac{2\times 3}{3^2+w^2} = \frac{6}{9+w^2}$$

$$= Cos(3w)\frac{6}{9+\left(\frac{w}{2}\right)^2}$$

$$= Cos(3w)\frac{6\times 4}{9\times 4+w^2}$$

$$= \frac{24}{36+w^2}Cos(3w)$$

(14.38) Determine the Fourier Transform of a signal $x(t) = 2e^{-3t}u(t) * 3e^{-2t}u(t)$

$$FT[x(t)] = FT[2e^{-3t}u(t) * 3e^{-2t}u(t)]$$

$$= FT[2e^{-3t}u(t)]\, FT[3e^{-2t}u(t)]$$

$$= 2FT[e^{-3t}u(t)]3FT[e^{-2t}u(t)]; FT[e^{-at}u(t)] = \frac{1}{a+jw}$$

$$= 2\left(\frac{1}{3+jw}\right)3\left(\frac{1}{2+jw}\right)$$

$$= \frac{6}{(3+jw)(2+jw)}$$

$$= \frac{6}{6-w^2+j5w}$$

(14.39) Determine the Fourier Transform of a signal y(t)=x(t)*x(-t), $given\ x(t) = 2e^{-3t}u(t)$

$$FT[y(t)] = FT[x(t) * x(-t)]$$

$$= FT[x(t)]\, FT[x(-t)]$$

$$= X(w)X(-w); where, X(w) = FT[x(t)] = FT[2e^{-3t}u(t)] = \frac{2}{3+jw}$$

$$= \left(\frac{2}{3+jw}\right)\left(\frac{2}{3-jw}\right)$$

$$= \frac{4}{9+w^2}$$

(14.40) Determine $X_1(w)*X_2(w)$, such that $x_1(t) = 2e^{-3t}u(t)\ and\ x_2(t) = 3e^{-2t}u(t)$

$$From\ frequency\ convolution\ theorem, FT[x_1(t)x_2(t)] = \frac{X_1(w) * X_2(w)}{2\pi}$$

$$\Rightarrow X_1(w) * X_2(w) = 2\pi FT[x_1(t)x_2(t)]$$

$$= 2\pi FT[2e^{-3t}u(t)3e^{-2t}u(t)]$$

$$= 2\pi FT[6e^{-5t}u(t)]$$

$$= 12\pi FT[e^{-5t}u(t)]$$

$$= 12\pi\left(\frac{1}{5+jw}\right)$$

$$= \frac{12\pi}{5+jw}$$

(14.41) Determine the Fourier Transform of a signal $y(t) = \frac{d}{dt}x(t), given\ x(t) = 2e^{-3t}u(t)$

$$FT[y(t)] = FT\left[\frac{d}{dt}x(t)\right]$$
$$= jw\ FT[x(t)]$$
$$= jw\ FT[2e^{-3t}u(t)]$$
$$= j2w\ FT[e^{-3t}u(t)]$$
$$= j2w\left(\frac{1}{3+jw}\right)$$
$$= \frac{j2w}{3+jw}$$

(14.42) Determine the Fourier Transform of a signal $y(t) = te^{-3t}u(t)$

$$FT[y(t)] = FT[te^{-3t}u(t)]; let\ e^{-3t}u(t) = x(t)$$
$$= FT[tx(t)]$$
$$= j\frac{d}{dw}X(w); X(w) = FT[x(t)] = FT[e^{-3t}u(t)] = \frac{1}{3+jw}$$
$$= j\ \frac{d}{dw}\left(\frac{1}{3+jw}\right)$$
$$= j\left(\frac{-1}{(3+jw)^2}(j)\right)$$
$$= \frac{1}{(3+jw)^2}$$
$$= \frac{1}{9-w^2+j6w}$$

(14.43) Determine the Fourier Transform of a signal $y(t) = t^2e^{-3t}u(t)$

$$FT[y(t)] = FT[t^2e^{-3t}u(t)]$$
$$= FT[t.te^{-3t}u(t)]; let\ te^{-3t}u(t) = x(t)$$
$$= FT[tx(t)]$$
$$= j\frac{d}{dw}X(w); X(w) = FT[x(t)] = FT[te^{-3t}u(t)] = \frac{1}{(3+jw)^2}$$
$$= j\ \frac{d}{dw}\left(\frac{1}{(3+jw)^2}\right)$$
$$= j\left(\frac{-2}{(3+jw)^3}(j)\right)$$
$$= \frac{2}{(3+jw)^3}$$

(14.44) Determine the Fourier Transform of a signal $y(t) = t^3 e^{-3t} u(t)$

$$FT[y(t)] = FT[t^3 e^{-3t} u(t)]$$
$$= FT[t.t^2 e^{-3t} u(t)]; Let\ t^2 e^{-3t} u(t) = x(t)$$
$$= FT[tx(t)]$$
$$= j\frac{d}{dw}X(w); X(w) = FT[x(t)] = FT[t^2 e^{-3t} u(t)] = \frac{2}{(3+jw)^3}$$
$$= j\ \frac{d}{dw}\left(\frac{2}{(3+jw)^3}\right)$$
$$= j\left(\frac{2(-3)}{(3+jw)^4}(j)\right)$$
$$= \frac{6}{(3+jw)^4}$$

(14.45) Find the Fourier Transform of $x(t) = 1$

We know that, $FT[\delta(t)] = 1$

Apply duality property, if $FT[x(t)] = X(w)$, *then* $FT[X(t)] = 2\pi x(-w)$

$$\Rightarrow FT[1] = 2\pi\delta(-w)$$
$$= 2\pi\delta(w)$$

(14.46) Find the Fourier Transform of $x(t) = u(t)$

$$FT[u(t)] = FT\left[\frac{1+Sgn(t)}{2}\right]$$
$$= \frac{1}{2}[FT[1] + FT[Sgn(t)]]; FT[1] = 2\pi\delta(w)\ \&\ FT[Sgn(t)] = \frac{2}{jw}$$
$$= \frac{1}{2}\left[2\pi\delta(w) + \frac{2}{jw}\right]$$
$$= \pi\delta(w) + \frac{1}{jw}$$

(14.47) Find the Fourier Transform of $x(t) = e^{jw_0 t}$

$$FT\left[e^{jw_0 t}\right] = FT\left[e^{jw_0 t}.1\right]; use\ frequency\ shifting\ property, FT\left[e^{jw_0 t}.x(t)\right] = X(w-w_0)$$
$$= X(w-w_0); where\ X(w) = FT[x(t)] = FT[1] = 2\pi\delta(w)$$
$$= 2\pi\delta(w-w_0)$$

(14.48) Find the Fourier Transform of $x(t) = e^{-jw_0t}$

$$FT[e^{-jw_0t}] = FT[e^{-jw_0t}.1]; use\ frequency\ shifting\ property, FT[e^{jw_0t}.x(t)] = X(w - w_0)$$
$$= X(w + w_0); where\ X(w) = FT[x(t)] = FT[1] = 2\pi\delta(w)$$
$$= 2\pi\delta(w + w_0)$$

(14.49) Find the Fourier Transform of $x(t) = Cos(w_0t)$

$$FT[Cos(w_0t)] = FT\left[\frac{e^{jw_0t} + e^{-jw_0t}}{2}\right]$$
$$= \frac{1}{2}(FT[e^{jw_0t}] + FT[e^{-jw_0t}]); use\ FT[e^{jw_0t}] = 2\pi\delta(w - w_0)$$
$$= \frac{1}{2}(2\pi\delta(w - w_0) + 2\pi\delta(w + w_0))$$
$$= \pi(\delta(w + w_0) + \delta(w - w_0))$$

(14.50) Find the Fourier Transform of $x(t) = Sin(w_0t)$

$$FT[Sin(w_0t)] = FT\left[\frac{e^{jw_0t} - e^{-jw_0t}}{2j}\right]$$
$$= \frac{1}{2j}(FT[e^{jw_0t}] - FT[e^{-jw_0t}]); FT[e^{jw_0t}] = 2\pi\delta(w - w_0)$$
$$= \frac{1}{2j}(2\pi\delta(w - w_0) - 2\pi\delta(w + w_0))$$
$$= -j\pi(\delta(w - w_0) - \delta(w + w_0))$$
$$= j\pi(\delta(w + w_0) - \delta(w - w_0))$$

(14.51) Find the Fourier Transform of $x(t) = Cos(w_0t)u(t)$

$$FT[Cos(w_0t)u(t)] = FT\left[\frac{e^{jw_0t} + e^{-jw_0t}}{2}u(t)\right]$$
$$= \frac{1}{2}(FT[e^{jw_0t}u(t)] + FT[e^{-jw_0t}u(t)])$$

$Apply\ frequency\ shifting\ property, FT[e^{jw_0t}.x(t)] = X(w - w_0)$

$$= \frac{1}{2}(X(w - w_0) + X(w + w_0)); X(w) = FT[u(t)] = \pi\delta(w) + \frac{1}{jw}$$
$$= \frac{1}{2}\left(\left(\pi\delta(w - w_0) + \frac{1}{j(w - w_0)}\right) + \left(\pi\delta(w + w_0) + \frac{1}{j(w + w_0)}\right)\right)$$

$$= \frac{\pi}{2}\big(\delta(w + w_0) + \delta(w - w_0)\big) + \frac{1}{2j}\left(\frac{1}{w - w_0} + \frac{1}{w + w_0}\right)$$

$$= \frac{\pi}{2}\big(\delta(w + w_0) + \delta(w - w_0)\big) + \frac{1}{2j}\left(\frac{2w}{w^2 - w_0^2}\right)$$

$$= \frac{\pi}{2}\big(\delta(w + w_0) + \delta(w - w_0)\big) - \frac{jw}{w^2 - w_0^2}$$

(14.52) Find the Fourier Transform of $x(t) = Sin(w_0 t)u(t)$

$$FT[Sin(w_0 t)u(t)] = FT\left[\frac{e^{jw_0 t} - e^{-jw_0 t}}{2j}u(t)\right]$$

$$= \frac{1}{2j}\left(FT\left[e^{jw_0 t}u(t)\right] - FT\left[e^{-jw_0 t}u(t)\right]\right)$$

$Apply\ frequency\ shifting\ property, FT\left[e^{jw_0 t}.x(t)\right] = X(w - w_0)$

$$= \frac{1}{2j}\big(X(w - w_0) - X(w + w_0)\big); X(w) = FT[u(t)] = \pi\delta(w) + \frac{1}{jw}$$

$$= \frac{1}{2j}\left(\left(\pi\delta(w - w_0) + \frac{1}{j(w - w_0)}\right) - \left(\pi\delta(w + w_0) + \frac{1}{j(w + w_0)}\right)\right)$$

$$= -j\frac{\pi}{2}\big(\delta(w - w_0) - \delta(w + w_0)\big) + \frac{1}{2j.j}\left(\frac{1}{w - w_0} - \frac{1}{w + w_0}\right)$$

$$= j\frac{\pi}{2}\big(\delta(w + w_0) - \delta(w - w_0)\big) - \frac{1}{2}\left(\frac{2w_0}{w^2 - w_0^2}\right)$$

$$= j\frac{\pi}{2}\big(\delta(w + w_0) - \delta(w - w_0)\big) - \frac{w_0}{w^2 - w_0^2}$$

(14.53) Find the Fourier Transform of $x(t) = \frac{1}{t}$

$We\ know\ that, FT[Sgn(t)] = \frac{2}{jw}$

$Apply\ duality\ property, if\ FT[x(t)] = X(w), then\ FT[X(t)] = 2\pi x(-w)$

$$\Rightarrow FT\left[\frac{2}{jt}\right] = 2\pi Sgn(-w)$$

$$\Rightarrow FT\left[\frac{1}{t}\right] = \frac{j}{2}(-2\pi Sgn(w))$$

$$= -j\pi Sgn(w)$$

(14.54) Find the Fourier Transform of $x(t) = \frac{1}{1+j2\pi t}$

$$We\ know\ that, FT[e^{-at}u(t)] = \frac{1}{a+jw} = \frac{1}{a+j2\pi f}$$

$$a = 1 \Rightarrow FT[e^{-t}u(t)] = \frac{1}{1+j2\pi f}$$

$$Apply\ duality\ property, if\ FT[x(t)] = X(f), then\ FT[X(t)] = x(-f)$$

$$\Rightarrow FT\left[\frac{1}{1+j2\pi t}\right] = e^{-(-f)}u(-f) = e^{f}u(-f)$$

(14.55) Find the Fourier Transform of $x(t) = Sa(t)$

$$We\ know\ that, FT\left[Arect\left(\frac{t}{T}\right)\right] = ATSa\left(\frac{wT}{2}\right)$$

$$Apply\ duality\ property, if\ FT[x(t)] = X(w), then\ FT[X(t)] = 2\pi x(-w)$$

$$\Rightarrow FT\left[ATSa\left(\frac{tT}{2}\right)\right] = 2\pi Arect\left(\frac{-w}{T}\right)$$

$$T = 2 \Rightarrow FT[2Sa(t)] = 2\pi rect\left(\frac{w}{2}\right)$$

$$\Rightarrow FT[Sa(t)] = \pi rect\left(\frac{w}{2}\right)$$

(14.56) Find the Fourier Transform of $x(t) = Sinc\left(\frac{t}{2}\right)$

$$We\ know\ that, FT\left[Arect\left(\frac{t}{T}\right)\right] = ATSinc(fT)$$

$$Apply\ duality\ property, if\ FT[x(t)] = X(f), then\ FT[X(t)] = x(-f)$$

$$\Rightarrow FT[ATSinc(tT)] = Arect\left(\frac{-f}{T}\right)$$

$$T = \frac{1}{2} \Rightarrow FT\left[\frac{A}{2}Sinc\left(\frac{t}{2}\right)\right] = Arect\left(\frac{f}{T}\right)$$

$$\Rightarrow FT\left[Sinc\left(\frac{t}{2}\right)\right] = 2rect(2f)$$

(14.57) Find the Fourier Transform of $x(t) = \frac{1}{1+t^2}$

$$We\ know\ that, FT\left[e^{-a|t|}\right] = \frac{2a}{a^2+w^2}$$

$$Apply\ duality\ property, if\ FT[x(t)] = X(w), then\ FT[X(t)] = 2\pi x(-w)$$

$$\Rightarrow FT\left[\frac{2a}{a^2+t^2}\right] = 2\pi e^{-a|-w|}, If\ a = 1$$

$$\Rightarrow FT\left[\frac{2}{1+t^2}\right] = 2\pi e^{-|w|} \Rightarrow FT\left[\frac{1}{1+t^2}\right] = \pi e^{-|w|}$$

(14.58) Find the Fourier Transform of $x(t) = \frac{1}{a^2+t^2}$

$$We\ know\ that, FT\left[e^{-a|t|}\right] = \frac{2a}{a^2+w^2}$$

$$Apply\ duality\ property, if\ FT[x(t)] = X(w), then\ FT[X(t)] = 2\pi x(-w)$$

$$\Rightarrow FT\left[\frac{2a}{a^2+t^2}\right] = 2\pi e^{-a|-w|}$$

$$\Rightarrow FT\left[\frac{1}{a^2+t^2}\right] = \frac{\pi}{a}e^{-a|w|}$$

(14.59) Find the Fourier Transform of $x(t) = \frac{36}{(4+9t^2)(9+4t^2)}$

$$x(t) = \frac{36}{(4+9t^2)(9+4t^2)}$$

$$= \frac{1}{\left(\frac{4}{9}+t^2\right)\left(\frac{9}{4}+t^2\right)}$$

$$= \frac{A}{\frac{4}{9}+t^2} + \frac{B}{\frac{9}{4}+t^2}; A = \frac{1}{\frac{9}{4}-\frac{4}{9}} = \frac{36}{65}\ \&\ B = \frac{1}{\frac{4}{9}-\frac{9}{4}} = -\frac{36}{65}$$

$$= \frac{36}{65}\left(\frac{1}{\left(\frac{2}{3}\right)^2+t^2} - \frac{1}{\left(\frac{3}{2}\right)^2+t^2}\right)$$

$$FT[x(t)] = \frac{36}{65}\left(FT\left[\frac{1}{\left(\frac{2}{3}\right)^2+t^2}\right] - FT\left[\frac{1}{\left(\frac{3}{2}\right)^2+t^2}\right]\right); use\ FT\left[\frac{1}{a^2+t^2}\right] = \frac{\pi}{a}e^{-a|w|}$$

$$= \frac{36}{65}\left(\frac{\pi}{\frac{2}{3}}e^{-\frac{2}{3}|w|} - \frac{\pi}{\frac{3}{2}}e^{-\frac{3}{2}|w|}\right)$$

$$= \frac{36\pi}{65}\left(\frac{3}{2}e^{-\frac{2}{3}|w|} - \frac{2}{3}e^{-\frac{3}{2}|w|}\right)$$

(14.60) Verify the Parsevall's theorem for a signal, $x(t) = 3e^{-2t}u(t)$

$$We\ know\ that, FT[x(t)] = X(w) = FT[3e^{-2t}u(t)] = \frac{3}{2 + jw}$$

$$Parsevall's\ Theorem, E = \int_{-\infty}^{\infty} |\,x(t)\,|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\,X(w)\,|^2\, dw$$

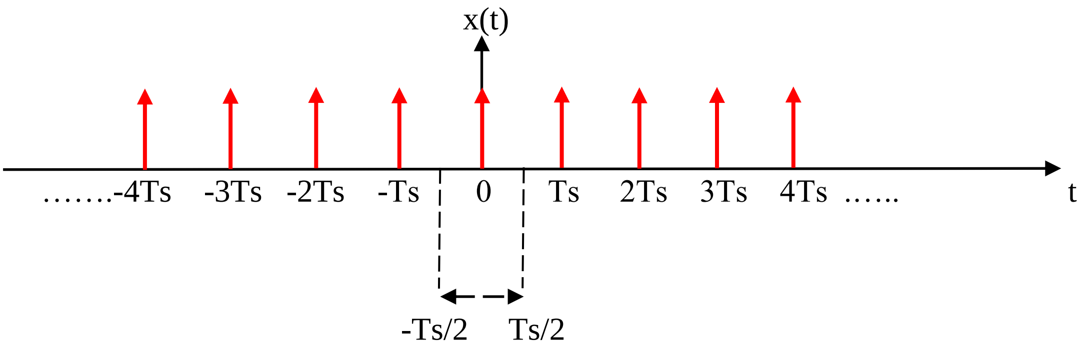
$$\Rightarrow LHS = \int_{-\infty}^{\infty} |\,3e^{-2t}u(t)|^2 dt = \int_{0}^{\infty} 9e^{-4t} dt = 9\frac{e^{-4t}}{-4}\Big|_{0}^{\infty} = \frac{9}{4}$$

$$\Rightarrow RHS = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{3}{2 + jw}\right|^2 dw = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{9}{4 + w^2} dw = \frac{9}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2^2 + w^2} dw$$

$$= \frac{9}{2\pi}\frac{1}{2}\tan^{-1}\frac{w}{2}\Big|_{-\infty}^{\infty} = \frac{9}{4\pi}(\tan^{-1}\infty - \tan^{-1}(-\infty)) = \frac{9}{4\pi}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{9}{4\pi}\pi = \frac{9}{4}$$

(14.61) Evaluate the Fourier Transform of a periodic train of impulse

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nTs) = \cdots + \delta(t + 2Ts) + \delta(t + Ts) + \delta(t) + \delta(t - Ts) + \delta(t - 2Ts) + \cdots$$



Fourier Transform of above periodic signal can be computed from the formula

$$FT[x(t)] = 2\pi \sum_{n=-\infty}^{\infty} C_n \delta(w - nw_0), w_0 = \frac{2\pi}{Ts} = 2\pi f_s\ = w_s$$

$$FT[x(t)] = 2\pi \sum_{n=-\infty}^{\infty} C_n \delta(w - nw_s)$$

Where,

$$C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jnw_0t} dt, x(t) = \delta(t)\ and\ T = Ts$$

$$= \frac{1}{Ts}\int_{-Ts/2}^{Ts/2} \delta(t)e^{-jnw_0t} dt\ , we\ know\ that\ \delta(t)x(t) = \delta(t)x(0)$$

$$= \frac{1}{Ts}\int_{-Ts/2}^{Ts/2} \delta(t)e^{0}dt$$

$$= \frac{1}{Ts}\int_{-Ts/2}^{Ts/2} \delta(t)dt$$

$$= \frac{1}{Ts}$$

$$\Rightarrow FT[x(t)] = 2\pi \sum_{n=-\infty}^{\infty} \frac{1}{Ts}\delta(w - nw_s)$$

$$= \frac{2\pi}{Ts} \sum_{n=-\infty}^{\infty} \delta(w - nw_s)$$

$$= 2\pi f_s \sum_{n=-\infty}^{\infty} \delta(w - nw_s)$$

$$= w_s \sum_{n=-\infty}^{\infty} \delta(w - nw_s)$$

$$\Rightarrow FT[x(t)] = w_s \sum_{n=-\infty}^{\infty} \delta(w - nw_s) = f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s) = \frac{1}{Ts} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{Ts}\right)$$

(14.62) Evaluate the Fourier Transform of a periodic signal x(t) = Cos(w_0t).

We know the Fourier Series representation of periodic signal

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnw_0t}$$

$$\Rightarrow Cos(w_0t) = \ldots + C_{-2}e^{-j2w_0t} + C_{-1}e^{-jw_0t} + C_0 + C_1e^{jw_0t} + C_2e^{j2w_0t} + \cdots$$

$$\Rightarrow \frac{e^{jw_0t} + e^{-jw_0t}}{2} = \frac{1}{2}e^{-jw_0t} + \frac{1}{2}e^{jw_0t} = C_{-1}e^{-jw_0t} + C_1e^{jw_0t}$$

$$\Rightarrow C_{-1} = \frac{1}{2}, C_1 = \frac{1}{2} \text{ and } C_n = 0, \text{other wise}$$

Fourier Transform of a periodic signal can be computed from the formula

$$FT[x(t)] = 2\pi \sum_{n=-\infty}^{\infty} C_n\delta(w - nw_0)$$

$$\Rightarrow FT[Cos(w_0t)] = 2\pi \sum_{n=-\infty}^{\infty} C_n\delta(w - nw_0)$$

$$= 2\pi\big(C_{-1}\delta(w + w_0) + C_1\delta(w - w_0)\big)$$

$$= 2\pi\left(\frac{1}{2}\delta(w + w_0) + \frac{1}{2}\delta(w - w_0)\right)$$

$$= \pi\big(\delta(w + w_0) + \delta(w - w_0)\big)$$

$$\Rightarrow FT[Cos(w_0t)] = \pi\big(\delta(w + w_0) + \delta(w - w_0)\big)$$

(14.63) Evaluate the Fourier Transform of a periodic signal x(t) = Sin(w_0t).

We know the Fourier Series representation of periodic signal

$$x(t) \ = \sum_{n=-\infty}^{\infty} C_n e^{jnw_0t}$$

$$\Rightarrow Sin(w_0t) = \frac{e^{jw_0t} - e^{-jw_0t}}{2j} = j\frac{1}{2}e^{-jw_0t} - j\frac{1}{2}e^{jw_0t} = C_{-1}e^{-jw_0t} + C_1e^{jw_0t}$$

$$\Rightarrow C_{-1} = j\frac{1}{2}, C_1 = -j\frac{1}{2} \ and \ C_n = 0, other\ wise$$

Fourier Transform of a periodic signal can be computed from the formula

$$FT[x(t)] = 2\pi \sum_{n=-\infty}^{\infty} C_n\delta(w - nw_0)$$

$$\Rightarrow FT[Sin(w_0t)] = 2\pi \sum_{n=-\infty}^{\infty} C_n\delta(w - nw_0)$$

$$= 2\pi\big(C_{-1}\delta(w + w_0) + C_1\delta(w - w_0)\big)$$

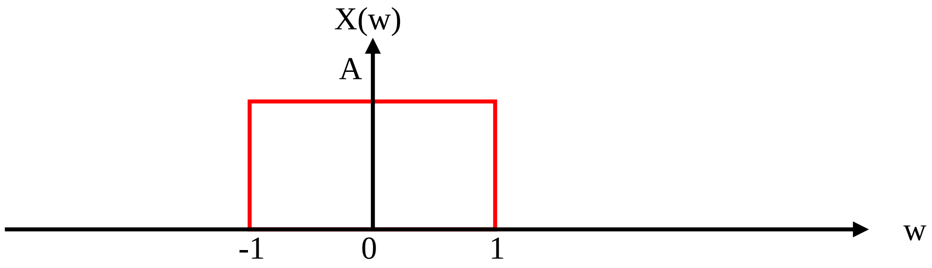
$$= 2\pi\left(j\frac{1}{2}\delta(w + w_0) - j\frac{1}{2}\delta(w - w_0)\right)$$

$$= j\pi\big(\delta(w + w_0) - \delta(w - w_0)\big)$$

$$\Rightarrow FT[Sin(w_0t)] = j\pi\big(\delta(w + w_0) - \delta(w - w_0)\big)$$

(14.64) Determine the Inverse Fourier Transform of rectangular function $X(w) = Arect\left(\frac{w}{2}\right)$.

$$We\ know, X(w) = A.rect\left(\frac{w}{2}\right) = \begin{cases} A, -1 < w < 1 \\ 0, Otherwise \end{cases}$$



From the definition of Inverse Fourier Transform

$$IFT[X(w)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)e^{jwt}dw$$

$$\Rightarrow IFT\left[Arect\left(\frac{w}{2}\right)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} Arect\left(\frac{w}{2}\right)e^{jwt}dw$$

$$= \int_{-1}^{1} Ae^{jwt}dw$$

$$= A\frac{e^{jwt}}{jt}\Bigg|_{-1}^{1}$$

$$= A\frac{e^{jt} - e^{-jt}}{jt}$$

$$= A\frac{2jSin(t)}{jt}$$

$$= 2A\frac{Sin(t)}{t}$$

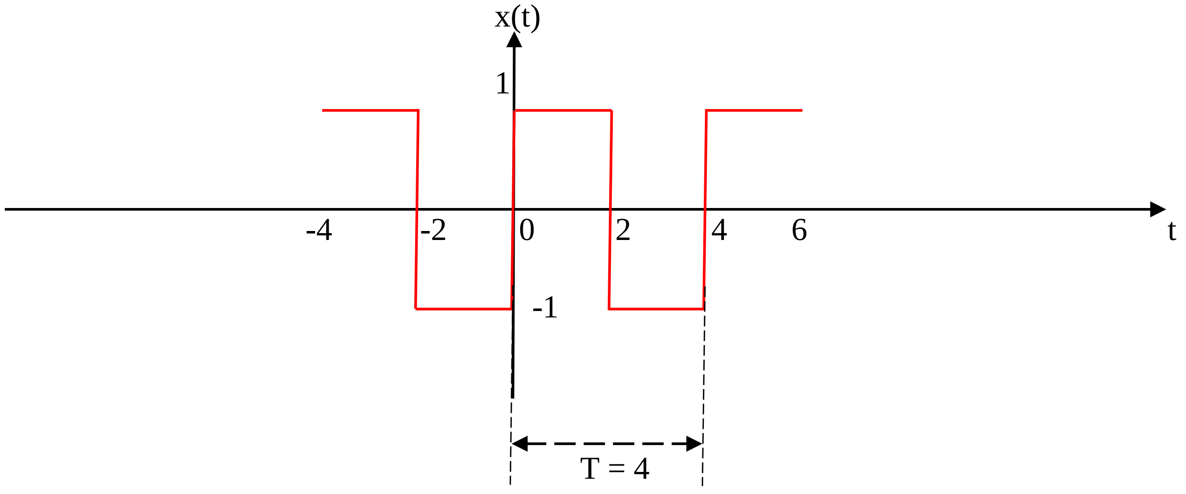
$$= 2ASa(t)$$

$$\Rightarrow IFT\left[Arect\left(\frac{w}{2}\right)\right] = 2ASa(t)$$

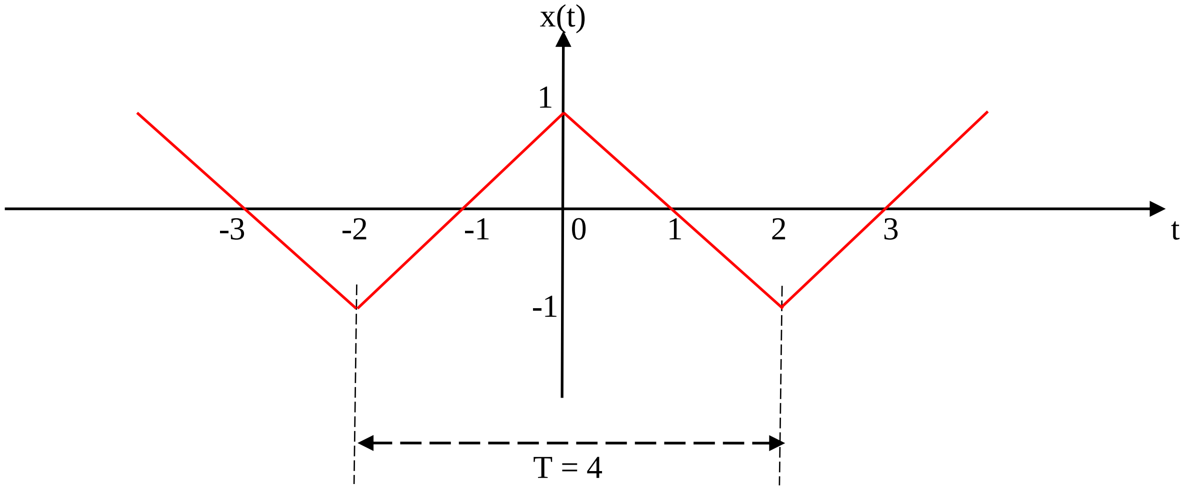
Note: Rectangular signal and Sampling signals are Fourier transformable pairs.

15. Assignment Questions

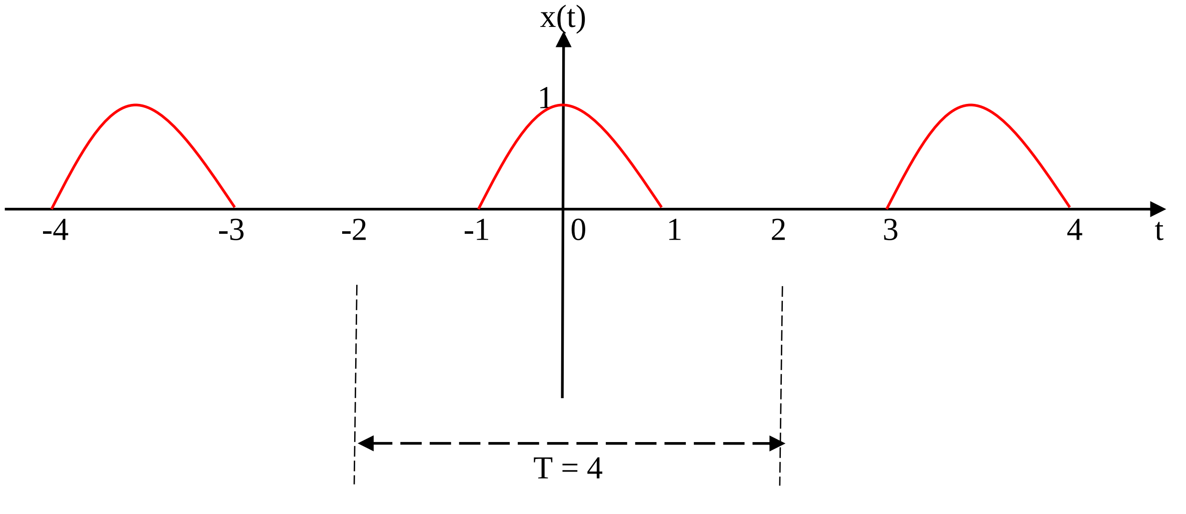
1. Determine the Trigonometric Fourier Series expansion of a periodic signal x(t) as shown



2. Determine the Trigonometric Fourier Series expansion of a periodic signal x(t) as shown

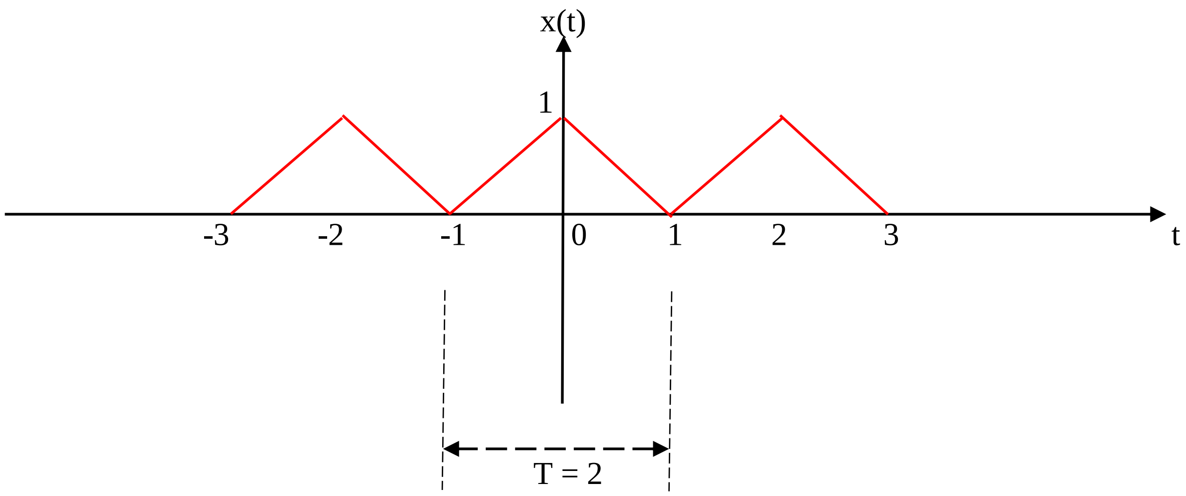


3. Determine the Exponential Fourier Series expansion of a periodic signal x(t) as shown

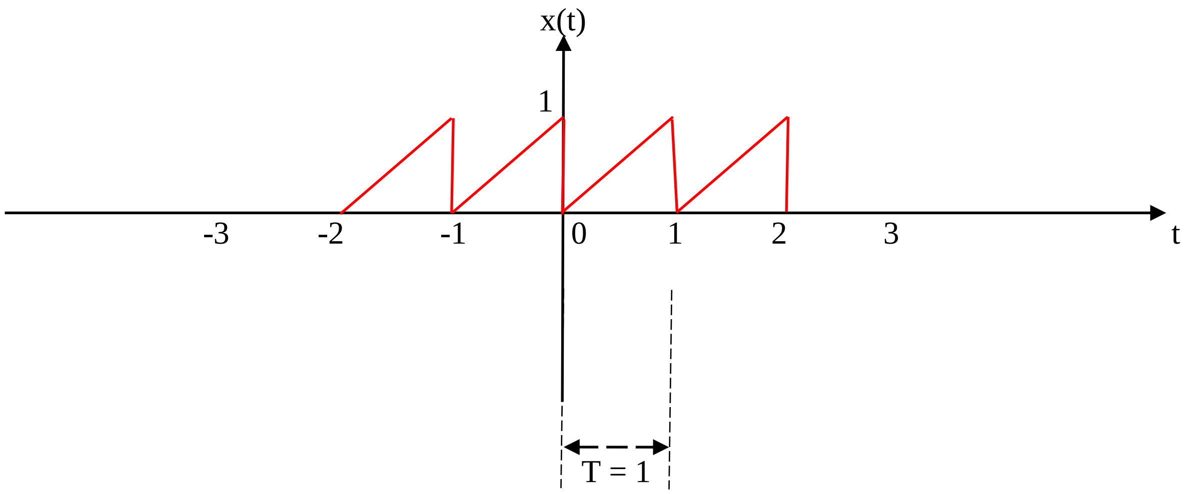


4. **Determine the Exponential Fourier Series representation for a continuous time periodic signal $x(t) = 1 - 2|t|, for|t| < 1$.**

5. **Determine the Exponential Fourier Series expansion of a periodic signal x(t) as shown**



6. **Determine the Exponential Fourier Series expansion of a periodic signal x(t) as shown**



7. **Evaluate the Fourier Transform of following aperiodic signals**
   a) $x(t) = \delta(t+1) + 2\delta(t) + \delta(t-1)$
   b) $x(t) = 2e^{-2(t+2)}u(t-2)$
   c) $x(t) = e^{-2|t-2|}$
   d) $x(t) = \frac{d}{dt}(u(-2-t) + u(t-2))$
   e) $x(t) = \frac{d^2}{dt^2}(\delta(2t-3))$

8. Evaluate the Fourier Transform of the following periodic signals

a) $x(t) = Sin(2\pi t + \frac{\pi}{4})$

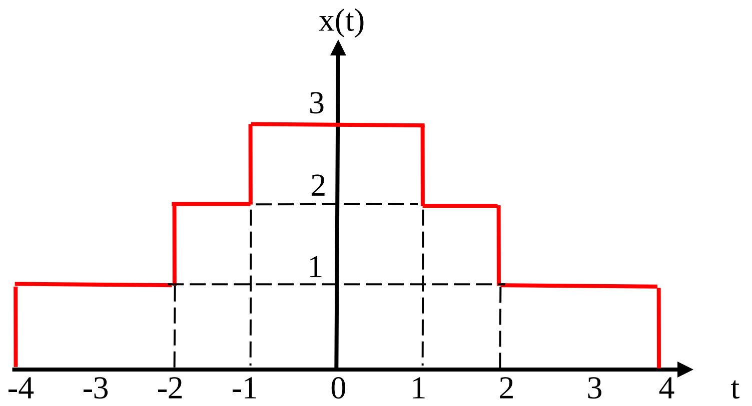
b) $x(t) = 1 + Cos(6\pi t + \frac{\pi}{8})$

c) $x(t) = 3Sin^2(2\pi t) + 2Sin^4(\pi t)$

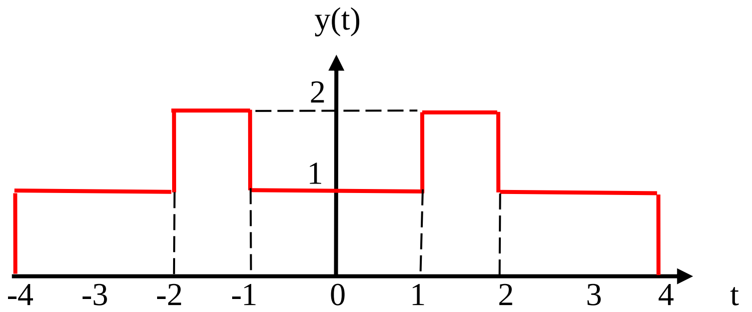
d) $x(t) = Sin^3(\pi t)$

9. Evaluate the Fourier Transform of the following waveforms

a) $x(t)$



b) $y(t)$



10. Evaluate the Inverse Fourier Transform of following aperiodic signals

a) $X(w) = 1$

b) $X(w) = 2\pi\delta(w) + \pi\delta(w - 4\pi) + \pi\delta(w + 4\pi)$

11. Let x(t) be a signal whose Fourier Tansform is $\boldsymbol{X(w) = \delta(w) + \delta(w - \pi) + \delta(w - 5)}$ and let $\boldsymbol{h(t) = u(t) - u(t - 2)}$

a) Is x(t) periodic?

b) Is h(t) periodic?

c) Is x(t)*h(t) periodic

12. Given the relationships $y(t) = x(t) * h(t)$ and $g(t) = x(3t) * h(3t)$

a) Obtain the relation between y(t) and g(t)

b) If the above relation is g(t)=Ay(kt), then find constants A and k.

13. Use appropriate properties to find Fourier Transform of

a) $x(t) = te^{-|t|}$

b) $x(t) = \frac{4t}{(1+t^2)^2}$

14. Evaluate the Inverse Fourier Transform of following aperiodic signals

a) $X(w) = \begin{cases} 1, & |w| < 1 \\ 0, & |w| > 1 \end{cases}$

b) $X(w) = \begin{cases} 2, & 0 \leq w \leq 2 \\ -2, & -2 \leq w < 0 \\ 0, & |w| > 2 \end{cases}$

15.**Given signal**

$$x(t) = \begin{cases} 0, & t < -\frac{1}{2} \\ t + \frac{1}{2}, & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 1, & t > \frac{1}{2} \end{cases}$$

a) Determine X(w) = FT[x(t)]

b) What is the Fourier Transform of $y(t) = x(t) - \frac{1}{2}$

16. Quiz Questions

(1) What is the expression for trigonometric Fourier series for x(t)?

(A) $x(t) = a_0 + \sum_{n=1}^{\infty}[a_n Cos(n\omega_0 t) + b_n Sin\,(n\omega_0 t)]$

(B) $x(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{jn\omega_0 t}$

(C) Both A and B

(D) None the above

(2) What is the expression for Exponential Fourier series for x(t)?

(A) $x(t) = a_0 + \sum_{n=1}^{\infty}[a_n Cos(\omega_0 nt) + b_n Sin\,(\omega_0 nt)]$

(B) $x(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega_0 t}$

(C) Both A and B

(D) None the above

(3) The effect of Gibb's phenomenon is getting ______ during the approximation of one signal with more no of orthogonal signals.

(A) Oscillations in pass band

(B) Oscillations in stop band

(C) Discontinuities at various time instants

(D) All the above

(4) What is the expression for exponential Fourier series coefficient?

(A) $C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jn\omega_0 t}dt$

(B) $C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{jn\omega_0 t}dt$

(C) $C_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)e^{-jn\omega_0 t}dt$

(D) $C_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)e^{jn\omega_0 t}dt$

(5) What is the expression for trigonometric Fourier series coefficient a_n?

(A) $a_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,cos(\,n\omega_0 t)dt$

(B) $a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\,cos(\,n\omega_0 t)dt$

(C) $a_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,sin(\,n\omega_0 t)dt$

(D) $a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\,sin(\,n\omega_0 t)dt$

(6) What is the expression for trigonometric Fourier series coefficient b_n?

(A) $b_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\cos(n\omega_0 t)dt$

(B) $b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(n\omega_0 t)dt$

(C) $b_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\sin(n\omega_0 t)dt$

(D) $b_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin(n\omega_0 t)dt$

(7) What is the expression for trigonometric Fourier series coefficient a_0?

(A) $a_0 = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)dt$

(B) $a_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)dt$

(C) $a_0 = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin(n\omega_0 t)dt$

(D) $a_0 = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(n\omega_0 t)dt$

(8) Which condition comes under existence of Fourier series?

(A) Over any period T, the signal x(t) is absolutely integrable, $\int_0^T |x(t)|dt < \infty$

(B) In any finite interval of time, the signal x(t) has only finite number of discontinuities. Furthermore, each of these discontinuities must be finite.

(C) In any finite interval of time, the signal x(t) has only finite number of maxima and minima.

(D) All the above

(9) How to represent C_n in terms of b_n and a_n?

(A) $C_n = \frac{a_n - jb_n}{2}$

(B) $C_n = \frac{a_n + jb_n}{2}$

(C) $C_n = \frac{-a_n - jb_n}{2}$

(D) $C_n = \frac{-a_n + jb_n}{2}$

(10) How to represent a_n in terms of C_n?

(A) $a_n = C_n + C_{-n}$

(B) $a_n = C_n - C_{-n}$

(C) $a_n = j(C_n + C_{-n})$

(D) $a_n = j(C_n - C_{-n})$

(11) How to represent b_n in terms of C_n?

(A) $b_n = C_n + C_{-n}$

(B) $\boldsymbol{b_n = C_n - C_{-n}}$

(C) $b_n = j(C_n + C_{-n})$

(D) $b_n = j(C_n - C_{-n})$

(12) Choose the correct symmetry condition with respect to Trigonometric Fourier series

(A) Odd signals have only sine terms

(B) Even signals have no sine terms

(C) Signals with half wave symmetry have only odd harmonics

(D) All the above

(13) Power spectrum is drawn between

(A) $\boldsymbol{|C_n|\ Vs\ w = nw_0}$

(B) $\boldsymbol{|C_n|^2\ Vs\ w = nw_0}$

(C) $|C_n|\ Vs\ t$

(D) $|C_n|^2\ Vs\ t$

(14) How the average power of a periodic signal can be computed from the coefficients of exponential Fourier series

(A) $\boldsymbol{P = \sum_{n=-\infty}^{\infty} |C_n{}^2|}$

(B) $\boldsymbol{P = 2\sum_{n=-\infty}^{\infty} |C_n{}^2|}$

(C) $P = |C_n{}^2|$

(D) $P = 2|C_n{}^2|$

If the Exponential and Trigonometric Fourier Series coefficients of x(t)=1+3Sin(2t)+2Cos3t are Cn, a_n and b_n, then

(15) Find $a_0 + C_0$

(A) 1 (B) 2 (C) 3 (D) 4

(16) Find b_1, b_2, b_3

(A) 0,3,0 (B) 3,0,0 (C) 0,0,3 (D) 0,3,2

(17) Find a_1, a_2, a_3

(A) 0,3,2 (B) 2,0,0 (C) 0,2,0 (D) 0,0,2

(18) Find $C_1 + C_{-1}$

(A) 1 (B) 2 (C) 3 (D) 0

(19) Find $C_2 + C_{-2}$

(A) 1 (B) 2 (C) 3 (D) 0

(20) Find $C_3 + C_{-3}$

(A) 1 (B) 2 (C) 3 (D) 0

(21) Find C_1C_{-1}

(A) 1 (B) 2 (C) 3 (D) 0

(22) Find $4C_2C_{-2}$

(A) 8 (B) 6 (C) 3 (D) 9

(23) Find C_3C_{-3}

(A) 1 (B) 2 (C) 3 (D) 0

(24) Find $C_1 + C_{-1} + C_2 + C_{-2} + C_3 + C_{-3}$

(A) 1 (B) 2 (C) 3 (D) 0

(25) Find the average power of x(t)

(A) 15 (B) 15/2 (C) 9/2 (D) 9

(26) Fourier transform is applicable to

(A) Periodic Signals

(B) Aperiodic Signals

(C) Both A & B

(D) None of the above

(27) The Fourier transform of a real valued signals has

(A) Odd Symmetry

(B) Even Symmetry

(C) Conjugate Symmetry

(D) Conjugate Anti Symmetry

(28) The Fourier transform of a Gaussian signal is

(A) Triangular signal

(B) Sampling signal

(C) Rectangular signal

(D) Gaussian signal

(29) The Fourier transform of a aperiodic signal x(t) can be obtained from

(A) $\boldsymbol{X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt}$

(B) $\boldsymbol{X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt}$

(C) $X(\omega) = \int_{-\infty}^{\infty} x(t)\boldsymbol{e^{j\omega t}}dt$

(D) Both (A) and (B)

(30) The Inverse Fourier transform is defined by

(A) $\boldsymbol{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega}$

(B) $\boldsymbol{x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df}$

(C) $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\boldsymbol{e^{-j\omega t}}d\omega$

(D) Both (A) and (B)

(31) The Fourier transform of a unit impulse signal δ(t) is

(A) $\frac{1}{j\omega}$

(B) $2\pi\delta(\omega)$

(C) 1

(4) $\delta(\omega)$

(32) The Fourier transform of a DC Signal x(t) = 1 is

(A) $\frac{1}{j\omega}$

(B) $2\pi\delta(\omega)$

(C) $\frac{1}{2\pi}\delta(\omega)$

(4) $\delta(\omega)$

(33) If the Fourier transform of x(t) is X(ω) then the Fourier transform of $x(t-t_0)$ is

(A) $e^{-j\omega t_0}X(\omega)$

(B) $e^{+j\omega t_0}X(\omega)$

(C) $X(\omega+\omega_0)$

(D) $X(\omega-\omega_0)$

(34) If the Fourier transform of x(t) is X(ω) then the Fourier transform of $x(t+t_0)$ is

(A) $e^{-j\omega t_0}X(\omega)$

(B) $e^{+j\omega t_0}X(\omega)$

(C) $X(\omega+\omega_0)$

(D) $X(\omega-\omega_0)$

(35) If the Fourier transform of x(t) is X(ω) then the Fourier transform of $e^{j\omega_0 t}x(t)$ is

(A) $e^{-j\omega t_0}X(\omega)$

(B) $e^{+j\omega t_0}X(\omega)$

(C) $X(\omega+\omega_0)$

(D) $X(\omega-\omega_0)$

(36) If the Fourier transform of x(t) is X(ω) then the Fourier transform of $e^{-j\omega_0 t}x(t)$ is

(A) $e^{-j\omega t_0}X(\omega)$

(B) $e^{+j\omega t_0}X(\omega)$

(C) $X(\omega+\omega_0)$

(D) $X(\omega-\omega_0)$

(37) If the Fourier transform of x(t) is X(ω) then the Fourier transform of $x(at)$ is

(A) $\frac{1}{a}X(a\omega)$

(B) $aX\left(\frac{\omega}{a}\right)$

(C) $\frac{1}{|a|}X\left(\frac{\omega}{a}\right)$

(D) $|a|X\left(\frac{\omega}{a}\right)$

(38) If the Fourier transform of x(t) is X(ω) then the Fourier transform of $\frac{d}{dt}x(t)$ is

(A) $j\omega X(\omega)$

(B) $\frac{X(\omega)}{j\omega}$

(C) $-j\int X(\omega)d\omega$

(D) $j\frac{d}{d\omega}X(\omega)$

(39) If the Fourier transform of x(t) is X(ω) then the Fourier transform of $\int x(t)dt$is

(A) $j\omega X(\omega)$

(B) $\frac{X(\omega)}{j\omega}$

(C) $-j\int X(\omega)d\omega$

(D) $j\frac{d}{d\omega}X(\omega)$

(40) If the Fourier transform of x(t) is X(ω) then the Fourier transform of $tx(t)$ is

(A) $j\omega X(\omega)$

(B) $\frac{X(\omega)}{j\omega}$

(C) $-j\int X(\omega)d\omega$

(D) $j\frac{d}{d\omega}X(\omega)$

(41) If the Fourier transform of x (t) is X (ω) then the Fourier transform of $\frac{x(t)}{t}$ is

(A) $j\omega X(\omega)$

(B) $\frac{X(\omega)}{j\omega}$

(C) $-j\int X(\omega)d\omega$

(D) $j\frac{d}{d\omega}X(\omega)$

(17) The Fourier transform of signum signal sgn(t) is

(A) $\frac{2}{j\omega}$

(B) $\pi\delta(\omega) + \frac{1}{j\omega}$

(C) $j\omega$

(D) $\frac{1}{1+j\omega}$

(42) The Fourier transform of u(t) function is

(A) $\pi\delta(\omega) + \frac{1}{j\omega}$

(B) $2\pi\delta(\omega) + \frac{1}{j\omega}$

(C) $\pi\delta(\omega) + \frac{2}{j\omega}$

(D) $\frac{1}{1+j\omega}$

(43) The Fourier transform of $Cos(\omega_0 t)$ is

(A) $\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$

(B) $\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$

(C) $j\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$

(D) $j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$

(44) The Fourier transform of $Sin(\omega_0 t)$ is

(A) $\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$

(B) $\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$

(C) $j\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$

(D) $j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$

(45) The Fourier transform of $e^{-at}u(t)$ is

(A) $\frac{1}{a-j\omega}$

(B) $\frac{1}{a+j\omega}$

(C) $\frac{1}{jw-a}$

(D) $\frac{-1}{a+j\omega}$

(46) The Fourier transform of $e^{at}u(-t)$ is

(A) $\frac{1}{a-j\omega}$

(B) $\frac{1}{a+j\omega}$

(C) $\frac{1}{jw-a}$

(D) $\frac{-1}{a+j\omega}$

(47) The Fourier transform of rectangular signal is

(A) Square signal

(B) Triangular signal

(C) Sinc signal

(D) Trapezoidal signal

(48) The Fourier transform of triangular signal is

(A) Rectangular signal

(B) Triangular signal

(C) Sinc square signal

(D) Trapezoidal signal

(49) If $x(t) = \delta(t-1) + \delta(t+1)$, then its Fourier transform is

(A) 0

(B) $2jSin(\omega)$

(C) $2Cos(\omega)$

(D) $Cos(\omega)$

(50) If $x(t) = e^{-2|t|}$, then its Fourier transform is

(A) $\frac{2}{2+\omega^2}$

(B) $\frac{4}{2+\omega^2}$

(C) $\frac{2}{4+\omega^2}$

(D) $\frac{4}{4+\omega^2}$